THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solution to Homework 7

1. Let S be a bounded sesquilinear form on $X \times Y$. Define

$$\|S\| \coloneqq \sup \{|S(x,y)| : \|x\| = 1,\ \|y\| = 1\}.$$

Show that

$$\|S\| = \sup \left\{ \frac{|S(x,y)|}{\|x\|\,\|y\|} : x \in X \setminus \{0\},\ y \in Y \setminus \{0\} \right\}$$

and

$$|S(x,y)| \leq \|S\|\,\|x\|\,\|y\|, \tag{1}$$

for all $x \in X$ and $y \in Y$.

Proof. Denote

$$\|S\|_* \coloneqq \sup \left\{ \frac{|S(x,y)|}{\|x\|\,\|y\|} : x \in X \setminus \{0\},\ y \in Y \setminus \{0\} \right\}.$$

For any $x \in X, y \in Y$ with $\|x\| = 1$ and $\|y\| = 1$, we have $\|S(x,y)\| = \dfrac{|S(x,y)|}{\|x\|\,\|y\|} \leq \|S\|_*$. Hence $\|S\| \leq \|S\|_*$. On the other hand by the sesquilinearity of S, for any $x \in X \setminus \{0\}$, $y \in Y \setminus \{0\}$,

$$\frac{|S(x,y)|}{\|x\|\,\|y\|} = |S(\frac{x}{\|x\|}, \frac{y}{\|y\|})| \leq \|S\|$$

where the last inequality holds since $x/\|x\|$ and $y/\|y\|$ are unit vectors, thus $\|S\|_* \leq \|S\|$. Together we have $\|S\| = \|S\|_*$.

Hence (1) holds for all $x \in X \setminus \{0\}$, $y \in Y \setminus \{0\}$. Since S is sesquilinear, we have

$$\begin{aligned} S(0,y) = S(0+0,y) = 2S(0,y) &\implies S(0,y) = 0 \\ S(x,0) = S(x,0+0) = 2S(x,0) &\implies S(x,0) = 0, \end{aligned}$$

thus (1) also holds when $x = 0 \in X$ or $y = 0 \in Y$. □

2. Let $T\colon \ell^2 \to \ell^2$ be defined by

$$T\colon (x_1, \ldots, x_n, \ldots) \mapsto (x_1, \ldots, \frac{1}{n}x_n, \ldots).$$

Show that the range $\mathcal{R}(T)$ is not closed in ℓ^2.

Proof. Suppose on the contrary that $\mathcal{R}(T)$ is closed in ℓ^2. Note that T is injective. It follows from Open Mapping Theorem that the map

$$S\colon \mathcal{R}(T) \to \ell^2,\ (y_1, \ldots, y_n, \ldots) \mapsto (y_1, \ldots, ny_n, \ldots)$$

is bounded. However, for $n \in \mathbb{N}$, let $e_n = (e_n(i))_{i=1}^\infty$ with $e_n(i) = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$. Then $e_n \in \mathcal{R}(T)$ and $\|e_n\| = 1$. Hence $\|S\| \geq \|Se_n\| = n \to \infty$ as $n \to \infty$, which contradicts the boundedness of S. □

3. Let T be a bounded operator on a complex Hilbert space H.

(a) Show that the operators

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint.

(b) Show that T is normal if and only if the operators T_1 and T_2 commute.

Proof. (a) For all $x, y \in H$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, T^{**}x\rangle} = \langle T^{**}x, y\rangle.$$

Hence $T^{**} = T$ since H is a complex Hilbert space. Then by $*\colon B(H) \to B(H)$ being a conjugate anti-isomorphism,

$$\begin{aligned}
T_1^* &= \left(\frac{1}{2}(T + T^*)\right)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T^* + T) = T_1, \\
T_2^* &= \left(\frac{1}{2i}(T - T^*)\right)^* = -\frac{1}{2i}(T^* - T^{**}) = -\frac{1}{2i}(T^* - T) = T_2.
\end{aligned}$$

(b) Since

$$\begin{aligned}
T_1T_2 &= \left(\frac{1}{2}(T + T^*)\right)\left(\frac{1}{2i}(T - T^*)\right) = \frac{1}{4i}(T^2 + T^*T - TT^* + (T^*)^2), \\
T_2T_1 &= \left(\frac{1}{2i}(T - T^*)\right)\left(\frac{1}{2}(T + T^*)\right) = \frac{1}{4i}(T^2 + TT^* - T^*T + (T^*)^2),
\end{aligned}$$

we have

$$T_1,\ T_2 \text{ commute} \iff T_1T_2 = T_2T_1 \iff T^*T = TT^* \iff T \text{ normal}.$$

□

— *THE END* —