

**THE CHINESE UNIVERSITY OF HONG KONG**  
**Department of Mathematics**  
**MATH4010 Functional Analysis 2021-22 Term 1**  
**Solution to Homework 6**

1. Prove that in a complex (resp. real) inner product space,  $x \perp y$  if and only if

$$\|x + \lambda y\| = \|x - \lambda y\| \tag{1}$$

for all scalars  $\lambda \in \mathbb{C}$  (resp.  $\mathbb{R}$ ).

*Proof.* Let  $X$  denote an inner product space with scalar field  $\mathbb{K}$ , where  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ .

( $\implies$ ) Let  $x, y \in X$ . If  $x \perp y$ , then  $x \perp \pm \lambda y$  for all  $\lambda \in \mathbb{K}$ . By Pythagorean theorem,

$$\|x + \lambda y\|^2 = \|x\|^2 + \|\lambda y\|^2 = \|x - \lambda y\|^2.$$

( $\impliedby$ ) It follows from Polarization identities (see e.g., [Tutorial 6, Theorem 2]) that for  $\forall x, y \in X$ , if  $\mathbb{K} = \mathbb{R}$ , then

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2), \tag{2}$$

and if  $\mathbb{K} = \mathbb{C}$ , then

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2). \tag{3}$$

Hence if  $\mathbb{K} = \mathbb{R}$ , then it follows from (2) that  $\langle x, y \rangle = 0$  by taking  $\lambda = 1$  in (1). If  $\mathbb{K} = \mathbb{C}$ , then it follows from (3) that  $\langle x, y \rangle = 0$  by taking  $\lambda = 1$  and  $i$  in (1).  $\square$

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