MATH2060B Mathematical Analysis II, Test I

Answer ALL Questions 20 Feb 2019, 10:30-11:15

1. (10 *ponits*) Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$
   Show that $f'(0)$ exists and $f'(x)$ does not exist for any $x \neq 0$.

   Proof: We first notice that we have

$$\frac{f(t+0)-f(0)}{t} = \begin{cases} t & t \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

   Hence, we have $f'(0) = \lim_{t\to 0} \frac{f(t+0)-f(0)}{t} = 0$.
   Now if $x \in \mathbb{Q} \setminus \{0\}$ and we take a sequence of irrational numbers (t_n) with $\lim t_n = 0$, then

$$\lim_{n\to\infty} \frac{f(t_n+x)-f(x)}{t_n} = \lim_{n\to\infty} \frac{0-x^2}{t_n}$$

   does not exist. So, $f'(x)$ does not exist if $x \in \mathbb{Q} \setminus \{0\}$.
   Finally, suppose that x is irrational. Let (t_n) be a sequence of non-zero rational numbers with $\lim t_n = 0$. Then the following limit does not exist:

$$\lim_{n\to\infty} \frac{f(t_n+x)-f(x)}{t_n} = \lim_{n\to\infty} \frac{(t_n+x)^2-0}{t_n}.$$

   Therefore, $f'(x)$ does not exist if x is irrational.
   The proof is finished.

2. (i) (5 *ponits*) Let f be a differentiable function on $(0, \infty)$. Show that if $f' > 0$ on $(0, \infty)$, then f is strictly increasing.

   Proof: Let $x, y \in (0, \infty)$ with $x < y$. The Mean Value Theorem tells us that there is an element $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Then by the assumption, we have $f(x) < f(y)$ as desired.

(ii) (10 *ponits*) Let $g : (0, \infty) \to \mathbb{R}$ be a differential function. Suppose that $g' < 0$ on $(0, \infty)$ and $g(c) > 0$ for some $c > 0$. Show that if $\lim_{x\to\infty} g(x) = 0$, then $g > 0$ on $(0, \infty)$.

Proof: By considering $-g$, Part (i) implies that g is a strictly decreasing function. So, if $g(a) \leq 0$ for some $a > 0$, then there exists a positive number $b > a$ such that $g(b) < 0$. Take $0 < \varepsilon < -g(b)$. Since $\lim_{x\to\infty} g(x) = 0$, we can find a point t with $t > b$ and $-\varepsilon < g(t)$ and thus, we have $g(b) < g(t)$. It leads to a contradiction because g is strictly decreasing.

(iii) (15 *ponits*) Let $f(x) := (1+\frac{1}{x})^x$ for $x \in (0, \infty)$. Show that f is a strictly increasing function on $(0, \infty)$.

Proof: Notice that $f(x) = \exp(x \ln(1 + \frac{1}{x}))$. So, we have

$$f'(x) = (1 + \frac{1}{x})^x[\ln(1 + \frac{1}{x}) - \frac{1}{x+1}].$$

If we put $g(x) := \ln(1 + \frac{1}{x}) - \frac{1}{x+1}$, for $x > 0$, then by Part (i), it suffices to show that $g > 0$ on $(0, \infty)$.
On the other hand,

$$g'(x) = \frac{-1}{x(x+1)^2} < 0$$

for all $x > 0$. Also, we have $\lim_{x\to\infty} g(x) = 0$ and $g(1) > 0$. Using Part (ii), we see that the function $g > 0$ on $(0, \infty)$ as desired. The proof is finished.

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