TA's solution to 2060B homework 9, 10

p.252 Q1. (2 marks)

The limit function is

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 2, \end{cases}$$

which is not continuous at $x = 1$. Since $(\dfrac{x^n}{1+x^n})$ is a sequence of continuous functions on $[0, 2]$, it cannot convergent uniformly on $[0, 2]$.

p.252 Q5. (4 marks)

Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$, $\exists \delta > 0$ s.t. for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \delta$. Then $\forall n \ge N$, $\forall x \in \mathbb{R}$, we have $\left|(x + \dfrac{1}{n}) - x\right| \le \dfrac{1}{N} < \delta$, so

$$|f_n(x) - f(x)| = \left|f(x + \frac{1}{n}) - f(x)\right| < \varepsilon.$$

This means f_n converges uniformly on $\mathbb{R}$ to f.

p.252 Q14. (4 marks)

The limit function f is given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \le 1. \end{cases}$$

Since $\left|f_n(\frac{1}{n}) - f(\frac{1}{n})\right| = \frac{1}{2}$ for all n, we see that the convergence is not uniform.

Since f_n is continuous, it is in $\mathcal{R}[0, 1]$. So is f, because it equals the constant function 1 on $[0, 1]$ except for a finite no. of points. It also follows that $\int_0^1 f = 1$. Finally, we have

$$\int_0^1 f_n = \int_0^1 \left(1 - \frac{1}{1+nx}\right) dx = 1 - \frac{\ln(1+n)}{n}.$$

By using L'Hospital's rule, or by noting that exponential growth is much faster than linear growth, the above is $\to 1$ when $n \to \infty$. The result follows.

p.280 Q1(c). The series is not convergent because $\lim_{n\to\infty} \dfrac{(-1)^{n+1} n}{n+2} \neq 0$. A fortiori, the series is not absolutely convergent.

p.280 Q1(d). We have the following well-written solution by previous work of this course (2012-13):

P.280, Q1(d)

Let $a_n := (-1)^{n+1} \dfrac{\ln n}{n}$.

Let $f(x) = \dfrac{\ln x}{x}$. Then $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x \geq 3$.

Hence the sequence $\left\{ \dfrac{\ln n}{n} \right\}$ is decreasing for $n \geq 3$.

Since $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$, by alternating test $\sum a_n$ is convergent.

And we have $|a_n| = \dfrac{\ln n}{n} > \dfrac{1}{n}$ for $n \geq 3$.

Since $\sum \dfrac{1}{n}$ is divergent, by comparison test $\sum |a_n|$ is divergent.

i.e. $\sum a_n$ is not absolutely convergent.

p.286 Q1(e). The following well-written solution is by previous work of this course (2012-13):

P.286, Q1(e)

It is clear that

$$f := \lim_n f_n = \begin{cases} 0 & \text{if } x < 1 \\ 1/2 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Hence $\sum f_n$ is clearly divergent for $x \geq 1$.

If $x < 1$, then $|f_n(x)| \leq x^n$.

Since $\sum x^n$ is convergent on $[0, 1)$, by comparison test $\sum f_n$ is convergent on $[0, 1)$.

Take $\epsilon := 1/4$.

For any m, n, take $x := (0.5)^{1/(n+1)}$.

Then we have

$$\begin{aligned} |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| &\geq f_{n+1}(x) \\ &= \frac{1}{3} \\ &> \epsilon \end{aligned}$$

So the convergence is not uniform on $[0, 1)$.

On the other hand, for any $M < 1$, we have $|f_n(x)| \leq x^n \leq M^n$ for $x \in [0, M]$.

Since $\sum M^n$ is convergent, by M-test the convergence is uniform on $[0, M]$.

An explanation for the highlighted line is as follows. For any $x_0 \geq 1$, $\exists N \in \mathbb{N}$ s.t. $f_n(x_0) > \dfrac{1}{3}$ for all $n \geq N$. Therefore,

$$\sum_1^\infty f_n(x_0) \geq \sum_N^\infty f_n(x_0) \geq \sum_N^\infty \frac{1}{3} = \infty.$$

p.286 Q1(f). The following well-written solution is by previous work of this course (2012-13):

P.286, Q1(f)
Let any $\epsilon > 0$ be given.
Take $N_1 \in \mathbb{N}$ s.t. $\forall x \geq 0,\ n \geq N_1$,

$$|f_n(x)| = \frac{1}{n} < \frac{\epsilon}{2}$$

For any $x \geq 0,\ n \in \mathbb{N}$, we have

$$\begin{aligned} |f_n(x) + f_{n+1}(x)| &= \left|(-1)^n \cdot \left(\frac{1}{n+x} - \frac{1}{n+1+x}\right)\right| \\ &= \left|\frac{1}{(n+x)(n+1+x)}\right| \\ &< \frac{1}{n^2} \end{aligned}$$

It is well known that $\sum \frac{1}{n^2}$ converges. Take $N_2 \in \mathbb{N}$ s.t. $\forall m, n \geq N_2$,

$$\left|\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2}\right| < \frac{\epsilon}{2}$$

Take $N := \max\{N_1, N_2\}$.
Then $\forall x \geq 0$ and $m, n \geq N$, in worst case if there are odd number of terms below,

$$\begin{aligned} |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| &\leq |f_{n+1}(x) + f_{n+2}(x)| + |f_{n+3}(x) + f_{n+4}(x)| \\ &\quad + \cdots + |f_{m-2}(x) + f_{m-1}(x)| + |f_m(x)| \\ &< \left|\frac{1}{(n+1)^2} + \frac{1}{(n+3)^2} + \cdots + \frac{1}{(m-2)^2}\right| + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

So $\sum f_n$ converges uniformly (and hence also pointwisely) for $x \geq 0$.

We only note that it should be "$\leq$" at the highlight point.

p.286 Q6(b). Since $t \mapsto t^\alpha$ is a continuous function and $\lim_{n\to\infty} \frac{n}{n+1} = 1$, we have

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^\alpha = 1.$$

Therefore, by formula, the answer is

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} (n+1) \left(\frac{n}{n+1}\right)^\alpha = \infty.$$

Note:

To check the answer imprecisely, observe that

$$\frac{n^\alpha}{n!} \lesssim \frac{n^\alpha}{\left(\frac{n}{2}\right)^{n/2} \cdot \left(\frac{n}{2}\right)!} \lesssim \frac{1}{\left(\frac{n}{2}\right)!},$$

therefore

$$\sum_n \frac{n^\alpha}{n!} |x|^n \lesssim 2 \sum_k \frac{1}{k!} \left|x^2\right|^k < \infty.$$

p.286 Q6(d). It follows from the formula $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ and L'Hospital's rule that the answer is 1.

Note:

To check the answer imprecisely, observe that when $x = 1$,

$$\sum \frac{1}{\ln n} \geq \sum \frac{1}{n} = \infty,$$

while when $x \in (0,1)$,

$$\sum \frac{x^n}{\ln n} \lesssim \sum x^n < \infty.$$

p.253 Q9. Since $||f_n||_{[0,1]} = \frac{1}{n} \to 0$, f is the zero function and the convergence is uniform.

For $x \in [0,1]$, when n tends to infinity, we have

$$f_n'(x) = x^{n-1} \to \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases}$$

The R.H.S. is the function g. Hence $g(1) = 1 \neq 0 = f'(1)$.