TA's solution to 2060B homework 8

p.246 Q4. (2 marks)

$$\lim_{n\to\infty}\frac{x^n}{1+x^n}=\begin{cases}0 & \text{if } 0\le x<1\\ \frac{1}{2} & \text{if } x=1\\ 1 & \text{if } 1<x.\end{cases}$$

p.246 Q14. (3 marks)

Let $f_n(x):=\dfrac{x^n}{1+x^n}$. For any $y\in[0,b]$, we have $|f_n(y)|\le|y^n|\le b^n$ and so $||f_n||_{[0,b]}\le b^n$. This implies $\lim_{n\to\infty}||f_n||_{[0,b]}=0$. Therefore (f_n) converges uniformly to the zero function on $[0,b]$.[*]

On the other hand, (f_n) does not converge uniformly on $[0,1]$. If it did, then (f_n) converges uniformly to the zero function on $[0,1)$.[†] Therefore $\lim_{n\to\infty}||f_n||_{[0,1)}=0$. But for any $n\in\mathbb{N}$, we have $\sqrt[n]{0.5}\in[0,1)$ and $\left|f_n(\sqrt[n]{0.5})\right|=\dfrac{1}{3}$, so $||f_n||_{[0,1)}\ge\dfrac{1}{3}$. This is a contradiction.

p.246 Q19. (3 marks)

Define $f_n:[0,\infty)\to\mathbb{R}$ by $f_n(x):=x^2e^{-nx}$.[‡] We have

$$f_n'(x)=2xe^{-nx}-x^2ne^{-nx}=x(2-xn)e^{-nx}.$$

By first derivative test, f_n attains absolute maximum at $\dfrac{2}{n}$, whence $||f_n||=\dfrac{4e^{-2}}{n^2}$. It follows that (f_n) converges uniformly to the zero function.

[*] We have used textbook 8.1.8 Lemma. Reading its proof carefully, we may skip the boundedness condition. This condition is for ensuring that $||g_n-g||_A$ exists in $\mathbb{R}$. However, if $|g_n-g|$ is unbounded on A for infinitely many n, then g_n cannot converge uniformly to g on A.

[†] "$f_n\rightrightarrows g$ on A" implies "$f_n\rightrightarrows g$ on B" for any nonempty $B\subseteq A$. Also, for $a\in A$ it implies "$\lim_{n\to\infty}f_n(a)=g(a)$". Since limit of sequence is unique, $g(a)$ is given by Q4.

[‡] Notice that the pointwise limit of f_n is the zero function. Then the greatest hindrance for f_n being not uniformly convergent is that for any n, there always is some x_n so that $|f_n(x_n)|$ keeps large. As a result, we want to study the maximum of $|f_n|$.

p.246 Q22. (2 marks)

Since $||f_n - f|| = \frac{1}{n}$, we see that (f_n) converges uniformly to f.[§] On the other hand, (f_n^2) does not converge uniformly on $\mathbb{R}$. If it did, then $|f_n^2(x) - f^2(x)| \to 0$ uniformly when $n \to \infty$.[¶] But for any $n \in \mathbb{N}$, we have

$$\left|f_n^2(n) - f^2(n)\right| = |f_n(n) - f(n)| \cdot |f_n(n) + f(n)| = \frac{2n + \frac{1}{n}}{n} > 2.$$

This is a contradiction.

[§]Again it is textbook 8.1.8 Lemma with the boundedness condition skipped.

[¶]By the same reason given in the Q14's footnote, the only possible uniform limit of (f_n^2) is f^2.