TA's solution to 2060B homework 7

p.225 Q21. (2+2+2+2 marks)

(a) Noting that $(tf \pm g)^2 \in \mathcal{R}[a,b]$ and $(tf \pm g)^2 \geq 0$ for all $t \in \mathbb{R}$, we have

$$\int_a^b (tf \pm g)^2 \geq \int_a^b 0 = 0.$$

(b) Note that $t^2 f^2$, $2tfg$, g^2 are all in $\mathcal{R}[a,b]$. Therefore, by linearity of Riemann integration,

$$\int_a^b (tf \pm g)^2 = \int_a^b (t^2 f^2 \pm 2tfg + g^2) = t^2 \int_a^b f^2 \pm 2t \int_a^b fg + \int_a^b g^2.$$

It follows from (a) that $\forall t > 0$,

$$t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2 \geq \pm 2 \int_a^b fg.$$

Hence

$$t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2 \geq \max\left\{2 \int_a^b fg, -2 \int_a^b fg\right\} = 2 \left| \int_a^b fg \right|.$$

(c) If $\int_a^b f^2 = 0$, then by (b), we have for all $t > 0$

$$\frac{1}{t} \int_a^b g^2 \geq 2 \left| \int_a^b fg \right|.$$

Letting $t \to \infty$ gives the desired result.

(d) By textbook/ lecture notes, we have $f \in \mathcal{R}[a,b] \Rightarrow |f| \in \mathcal{R}[a,b]$ and $0 \leq \left| \int fg \right| \leq \int |fg|$. Taking square we get one of the desired results.

On the other hand, by the calculation in (a) and (b), we have $\forall t \in \mathbb{R}$,

$$t^2 \int_a^b |f|^2 + 2t \int_a^b |f|\,|g| + \int_a^b |g|^2 \geq 0.$$

By the theory of quadratic equation, this means its discriminant is ≤ 0, so

$$\left(2\int_a^b |fg|\right)^2 - 4\left(\int_a^b f^2\right)\cdot\left(\int_a^b g^2\right) \leq 0.$$

The result follows.

p.225 Q22. (2 marks)

$h : [0,1] \to \mathbb{R}$ is defined by

$$h(x) := \begin{cases} \frac{1}{s} & \text{if } x = \frac{r}{s} \in \mathbb{Q}, \text{ where } r, s \in \mathbb{N} \text{ and } \gcd(r,s) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{s} & \text{if } x = \frac{r}{s} \in \mathbb{Q}, \text{ where } s \in \mathbb{N}, r \in \mathbb{N} \cup \{0\} \text{ and } \gcd(r,s) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\operatorname{sgn} \circ h(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

which is not in $\mathcal{R}[0,1]$ because

$$\overline{\int_0^1} \operatorname{sgn} \circ h = 1 \quad \text{while} \quad \underline{\int_0^1} \operatorname{sgn} \circ h = 0.$$