TA's solution to 2060B homework 2

p.179 Q13. (3 marks)

Let $a < b$ be two points in I. Since f is differentiable on I, f is continuous on $[a, b]$ and differentiable on (a, b). Therefore, by Mean Value Theorem, $\exists$? $\in (a, b)$ such that

$$f(b) - f(a) = f'(?)(b - a).$$

By assumption f' is positive on I, so the above gives $f(b) - f(a) > 0$. As $a < b$ are arbitrary points in I, we conclude that f is strictly increasing on I.

p.179 Q15. (3 marks)

By assumption, $\exists M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$.

Now let $a < b$ be in I. Since f is differentiable on I, f is continuous on $[a, b]$ and differentiable on (a, b). Therefore, by Mean Value Theorem, $\exists$? $\in (a, b)$ such that

$$f(b) - f(a) = f'(?)(b - a).$$

As ? $\in (a, b) \subseteq I$, it follows that

$$|f(b) - f(a)| = |f'(?)| \, |b - a| \leq M \, |b - a| \, .$$

On the other hand, when $a = b$ the above holds plainly. Therefore f satisfies a Lipschitz condition on I.

p.187 Q5. (4 marks)

Note that

$$\frac{f(x)}{g(x)} = \frac{x^2 \sin(\frac{1}{x})}{\sin x} = \frac{x}{\sin x} \cdot x \sin(\frac{1}{x})$$

and that $\lim_{x \to 0} \frac{x}{\sin x} = 1$, $\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$, so we have*

$$\begin{aligned} \lim_{x \to 0} \frac{f(x)}{g(x)} &= \lim_{x \to 0} \left[\frac{x}{\sin x} \right] \cdot \lim_{x \to 0} \left[x \sin(\frac{1}{x}) \right] \\ &= 0. \end{aligned}$$

*This is textbook 4.2.4 Theorem. In view of the "if-then" logic flow, an important point is that we have to ensure $\lim \frac{x}{\sin x}$ and $\lim x \sin(\frac{1}{x})$ exist first before writing $\lim \frac{f(x)}{g(x)} = \lim \left[\frac{x}{\sin x} \right] \cdot \lim \left[x \sin(\frac{1}{x}) \right]$.

On the other hand, we have $\frac{1}{2n\pi} \to 0$, $\frac{1}{(2n+1)\pi} \to 0$ as $n \to \infty$, and

$$\begin{aligned}
\lim_{n\to\infty} \frac{f'(\frac{1}{2n\pi})}{g'(\frac{1}{2n\pi})} &= \lim_{n\to\infty} \left[\frac{2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})}{\cos x}\right]\Bigg|_{x=1/2n\pi} \\
&= \lim_{n\to\infty} \frac{0-1}{\cos(\frac{1}{2n\pi})} \\
&= -1,
\end{aligned}$$

while

$$\lim_{n\to\infty} \frac{f'(\frac{1}{(2n+1)\pi})}{g'(\frac{1}{(2n+1)\pi})} = \lim_{n\to\infty} \frac{0-(-1)}{\cos(\frac{1}{(2n+1)\pi})} = 1.$$

This shows that $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ does not exist.

Remark:

The following argument is an inappropriate use of textbook 4.2.4 Theorem. The reason has been given in the footnote.

"Suppose $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ exists. Then

$$\begin{aligned}
\lim_{x\to 0} \frac{2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})}{\cos x} &= \lim_{x\to 0} \frac{2x\sin(\frac{1}{x})}{\cos x} - \lim_{x\to 0} \frac{\cos(\frac{1}{x})}{\cos x} \\
&= 0 - \frac{\lim_{x\to 0}\cos(\frac{1}{x})}{\lim_{x\to 0}\cos x} \\
&= -\lim_{x\to 0} \cos(\frac{1}{x}).
\end{aligned}$$

So $\lim_{x\to 0}\cos(\frac{1}{x})$ exists. But then ... , which is a contradiction."