

# Real Analysis 20-12-04

## § 5.2 Radon-Nikodym Thm

Def (Absolute continuity).

Let  $\mu$  be a measure on  $(X, \mathcal{M})$ .

Let  $\lambda$  be a measure / signed measure on  $(X, \mathcal{M})$ .

We say that  $\lambda$  is abs. cts w.r.t  $\mu$  if

$$E \in \mathcal{M}, \mu(E) = 0 \Rightarrow \lambda(E) = 0.$$

We write  $\lambda \ll \mu$ .

Def. • Say that  $\lambda$  is concentrated on  $A \in \mathcal{M}$

$$\text{if } \lambda(E) = \lambda(E \cap A) \quad \forall E \in \mathcal{M}.$$

• Say that  $\lambda_1 \perp \lambda_2$  if  $\exists$  disjoint  $A, B \in \mathcal{M}$

such that  $\lambda_1$  is concentrated on  $A$ ,

$\lambda_2$  is concentrated on  $B$ .

Prop 5.3. Let  $\mu$  be a measure on  $(X, \mathcal{M})$

Let  $\lambda_1, \lambda_2$  be measures / signed measures on  $(X, \mathcal{M})$ .

Then the following hold:

(a)  $\lambda$  is concentrated on  $A \in \mathcal{M}$

$\Rightarrow |\lambda|$  is concentrated on  $A \in \mathcal{M}$

(b)  $\lambda_1 \perp \lambda_2 \Rightarrow |\lambda_1| \perp |\lambda_2|$

(c)  $\lambda_1 \perp \mu, \lambda_2 \perp \mu \Rightarrow \lambda_1 + \lambda_2 \perp \mu$

(d)  $\lambda_1 \ll \mu, \lambda_2 \ll \mu \Rightarrow \lambda_1 + \lambda_2 \ll \mu$ .

(e)  $\lambda \ll \mu \Rightarrow |\lambda| \ll \mu$

(f)  $\lambda_1 \ll \mu, \lambda_2 \perp \mu \Rightarrow \lambda_1 \perp \lambda_2$

(g)  $\lambda \ll \mu, \lambda \perp \mu \Rightarrow \lambda = 0$ .

Pf of (g): Since  $\lambda \perp \mu$ ,  $\exists$  disjoint  $A, B \in \mathcal{M}$

for which  $\lambda$  is concentrated on  $A$ ,  $\mu$  is concentrated on  $B$ .

Hence  $\forall E \in \mathcal{M}$ ,

$$\lambda(E) = \lambda(E \cap A).$$

In the mean time,

$$\mu(E \cap A) = \mu(E \cap A \cap B) = 0.$$

Since  $\lambda \ll \mu$ ,  $\lambda(E \cap A) = 0$

Hence  $\lambda(E) = \lambda(E \cap A) = 0$ .  $\square$

### Thm 5.5 (Lebesgue decomposition)

Let  $\mu$  be a  $\sigma$ -finite measure on  $(X, \mathcal{M})$ .

Let  $\lambda$  be a signed measure on  $(X, \mathcal{M})$ .

Then  $\exists$  a unique decomposition

$$\lambda = \lambda_{ac} + \lambda_s$$

Such that  $\lambda_{ac} \ll \mu$  and  $\lambda_s \perp \mu$ .

### Thm 5.6 (Radon-Nikodym Thm).

Under the same assumptions in Thm 5.5.

Suppose that  $\lambda \ll \mu$ .

Then  $\exists$  a unique  $h \in L^1(\mu)$  such that

$$\lambda(E) = \int_E h d\mu, \quad \forall E \in \mathcal{M}.$$

Pf of Thm 5.5 & Thm 5.6.

Step 1. We first consider the case when both  $\mu$  and  $\lambda$  are finite measures.

Let  $\rho = \mu + \lambda$ . Then  $\rho$  is a finite measure. Next we introduce  $\Lambda: L^2(\rho) \rightarrow \mathbb{R}$  by

$$\textcircled{1} \quad \Lambda(\phi) = \int \phi d\lambda, \quad \forall \phi \in L^2(\rho).$$

By Cauchy-Schwartz inequality,

$$\begin{aligned} |\Lambda(\phi)| &= \left| \int \phi d\lambda \right| \\ &\leq \left( \int \phi^2 d\lambda \right)^{1/2} \left( \int 1 d\lambda \right)^{1/2} \\ &\leq \left( \int \phi^2 d\rho \right)^{1/2} \lambda(X)^{1/2} \end{aligned}$$

$$= \|\phi\|_{L^2(\rho)} \cdot \lambda(X)^{1/2}$$

Hence  $\Lambda \in L^2(\rho)'$

Hence  $\exists g \in L^2(\rho)$  such that

$$(2) \quad \Lambda(\phi) = \int \phi g \, d\rho, \quad \forall \phi \in L^2(\rho) \\ = \int \phi \, d\lambda$$

Taking  $\phi = \chi_E$  in (2) gives

$$\Lambda(\chi_E) = \int \chi_E g \, d\rho = \int_E g \, d\rho \\ = \int \chi_E \, d\lambda = \lambda(E).$$

That is,

$$\lambda(E) = \int_E g \, d\rho, \quad \forall E \in \mathcal{M}$$

$$\text{So } 0 \leq \frac{1}{\rho(E)} \int_E g \, d\rho = \frac{\lambda(E)}{\rho(E)} \leq 1, \quad \forall E \in \mathcal{M}.$$

This implies  $0 \leq g(x) \leq 1$  for  $\rho$ -a.e  $x$ .

Suppose on the contrary that  $\exists \varepsilon > 0$  such that  
either ①  $g > 1 + \varepsilon$  on a set  $C$   
with  $p(C) > 0$ .

or ②  $g < -\varepsilon$  on a set  
 $\tilde{C}$  with  $p(\tilde{C}) > 0$ .

If ① holds,

$$\frac{1}{p(C)} \int_C g \, d\rho \geq 1 + \varepsilon \quad \text{leading to a contradiction}$$

If ② holds

$$\frac{1}{p(\tilde{C})} \int_{\tilde{C}} g \, d\rho \leq -\varepsilon, \quad \text{a contradiction}$$

Now by redefining  $g$  on a null set

we may assume

$$0 \leq g(x) \leq 1, \quad \forall x \in X.$$

Then we define

$$A = \{x : g(x) \in [0, 1)\},$$

$$B = \{x : g(x) = 1\}.$$

Define for  $E \in \mathcal{M}$ ,

$$\lambda_{ac}(E) = \lambda(E \cap A)$$

$$\lambda_s(E) = \lambda(E \cap B).$$

By definition,  $\lambda_{ac}$  is concentrated on  $A$   
 $\lambda_s$  is concentrated on  $B$ .

Recall that

$$\begin{aligned} \int \phi d\lambda &= \int \phi g d\rho, \quad \forall \phi \in L^2(\rho) \\ &= \int \phi g d\lambda + \int \phi g d\mu \end{aligned}$$

We have

$$\textcircled{3} \int \phi(1-g) d\lambda = \int \phi g d\mu.$$

Taking  $\phi = \chi_B$  in  $\textcircled{3}$  gives

$$\int_B 1-g d\lambda = \int_B g d\mu$$

Hence  $0 = \mu(B)$ .

Recall that  $\lambda_S$  is concentrated on  $B$ ,  
but  $\mu$  is concentrated on  $X \setminus B$

So  $\lambda_S \perp \mu$ .

Next we prove  $\lambda_{ac} \ll \mu$ .

To see this, taking  $\phi = \chi_E (1 + g + \dots + g^n)$  in (3)  
gives

$$\int \chi_E (1 + g + \dots + g^n) (1 - g) d\lambda = \int \chi_E (1 + g + \dots + g^n) g d\mu$$

$$\text{LHS} = \left( \int_A + \int_B \right) \chi_E (1 + g + \dots + g^n) (1 - g) d\lambda$$

$$= \int_{A \cap E} (1 + g + \dots + g^n) (1 - g) d\lambda$$

$$= \int_{A \cap E} 1 - g^{n+1} d\lambda$$

$$\longrightarrow \int_{A \cap E} 1 d\lambda = \lambda(E \cap A) = \lambda_{ac}(E)$$

as  $n \rightarrow \infty$ .



$$(RHS) = \left( \int_A + \int_B \right) \chi_E (1 + g + \dots + g^n) g \, d\mu$$

$$\stackrel{\mu(B)=0}{=} \int_{A \cap E} g \cdot \frac{1 - g^{n+1}}{1 - g} \, d\mu$$

$$\longrightarrow \int_{A \cap E} \frac{g}{1 - g} \, d\mu \quad \text{as } n \rightarrow \infty$$

$$= \int_E \frac{g}{1 - g} \, d\mu \quad (\text{since } \mu(B) = 0)$$

Hence  $\lambda_{ac}(E) = \int_E \frac{g}{1 - g} \, d\mu, \quad \forall E \in \mathcal{M}.$

Set  $h = \frac{g}{1 - g}$ . Then

$$\lambda_{ac}(E) = \int_E h \, d\mu$$

which implies  $\infty > \lambda_{ac}(X) = \int_X h \, d\mu,$

Hence  $h \in L^1(\mu)$  and

$$\lambda_{ac}^{(E)} = \int_E h \, d\mu \Rightarrow \lambda_{ac} \ll \mu.$$

Step 2. Assume  $\mu$  is finite,  $\lambda$  is a signed measure.

Define

$$\lambda^+ = \frac{1}{2}(\lambda + |\lambda|),$$

$$\lambda^- = \frac{1}{2}(|\lambda| - \lambda).$$

Then from the property that

$$|\lambda(E)| \leq |\lambda|(E), \quad E \in \mathcal{M}$$

We see that

$\lambda^+, \lambda^-$  are two finite measures on  $(X, \mathcal{M})$ .

Moreover  $\lambda = \lambda^+ - \lambda^-$ .

We call the above decomposition the **Jordan decomposition** of  $\lambda$ .

By step 1,

$$\lambda^+ = \lambda_{ac}^+ + \lambda_s^+$$

$$\lambda^- = \lambda_{ac}^- + \lambda_s^-$$

where  $\lambda_{ac}^+ \ll \mu$ ,  $\lambda_{ac}^- \ll \mu$ ,  $\lambda_s^+, \lambda_s^- \perp \mu$ .

Hence

$$\begin{aligned}\lambda &= \lambda^+ - \lambda^- \\ &= (\lambda_{ac}^+ - \lambda_{ac}^-) + (\lambda_s^+ - \lambda_s^-)\end{aligned}$$

Now

$$\lambda_{ac}^+ - \lambda_{ac}^- \ll \mu, \quad \lambda_s^+ - \lambda_s^- \perp \mu.$$

Step 3. Now consider the case that  $\mu$  is  $\sigma$ -finite and  $\lambda$  is a signed measure.

Let  $\{X_j\}_{j=1}^{\infty}$  be a partition of  $X$  such that

$$\mu(X_j) < \infty.$$

Write

$$\mu_j = \mu|_{X_j}$$

$$\lambda_j = \lambda|_{X_j}$$

(i.e.  $\mu_j(E) = \mu(E \cap X_j)$ ,  $\lambda_j(E) = \lambda(E \cap X_j)$ )

Then  $\mu_j$  are finite measures on  $(X, \mathcal{M})$   
 $\lambda_j$  are signed measures on  $(X, \mathcal{M})$ .

$$\text{Then } \lambda_j = \lambda_{ac}^j + \lambda_s^j$$

$$\lambda_{ac}^j \ll \mu_j$$

$$\lambda_s^j \perp \mu_j$$

Then  $\exists h_j \in L^1(\mu_j)$  s.t

$$\lambda_{ac}^j(E) = \int_E h_j d\mu_j, \quad j=1, \dots$$

Finally let

$$\lambda_{ac} = \sum_{j=1}^{\infty} \lambda_{ac}^j$$

$$\lambda_s = \sum_{j=1}^{\infty} \lambda_s^j$$

Then  $\lambda_{ac} \ll \mu$  and  $\lambda_s \perp \mu$

$$\lambda_{ac}(E) = \int_E h d\mu,$$

where

$$h = \sum_{j=1}^{\infty} p_j \chi_{x_j} .$$

Step 4. (Uniqueness of the Lebesgue decomposition)

$\mu$  —  $\sigma$ -finite measure

$\lambda$  — a signed measure.

Suppose

$$\lambda = \lambda_1 + \lambda_2$$

$$= \lambda_3 + \lambda_4$$

such that  $\lambda_1, \lambda_3 \ll \mu$

$\lambda_2, \lambda_4 \perp \mu$ .

Notice that  $\lambda_1 - \lambda_3 \ll \mu$

But  $\lambda_1 - \lambda_3 = \lambda_4 - \lambda_2 \perp \mu$

By (8) of Prop 5.3,  $\lambda_1 - \lambda_3 = 0$

Hence  $\lambda_1 = \lambda_3$ ,  $\lambda_2 = \lambda_4$ .  $\square$

Uniqueness part in R-N Thm.

Suppose  $\lambda \ll \mu$  and  $\exists h_1, h_2 \in L^1(\mu)$

such that

$$\lambda(E) = \int_E h_1 d\mu = \int_E h_2 d\mu, \quad \forall E \in \mathcal{M}$$

Hence

$$\int_E h_1 - h_2 d\mu = 0, \quad \forall E \in \mathcal{M}.$$

We need to show that  $h_1 - h_2 = 0$  a.e.

If not, <sup>then</sup>  $\exists \varepsilon > 0$  such that either

$h_1 - h_2 > \varepsilon$  on a set  $B$  of positive measure  
or  $h_1 - h_2 < -\varepsilon$  on a set  $B$  of positive meas.

$$\int_B h_1 - h_2 d\mu \geq \sum \mu(B) > 0$$

leading a contradiction

Prop 5.7. Let  $\mu$  be a signed measure on  $(X, \mathcal{M})$ .

Let  $|\mu|$  denote the total variation of  $\mu$ .

Then the following hold:

①  $\exists h \in L^1(|\mu|)$  such that  $|h|=1$  for  $|\mu|$ -a.e.

and

$$\mu(E) = \int_E h d|\mu|.$$

②  $\exists$  disjoint  $A, B \in \mathcal{M}$  such that

$$\mu^+(E) = \mu(E \cap A), \quad \forall E \in \mathcal{M}$$

$$\mu^-(E) = -\mu(E \cap B), \quad \forall E \in \mathcal{M},$$

where  $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ ,  $\mu^- = \frac{1}{2}(|\mu| - \mu)$ .

③ If  $\mu = \lambda_1 - \lambda_2$  for two measures  $\lambda_1, \lambda_2$   
then  $\lambda_1 \geq \mu^+$ ,  $\lambda_2 \geq \mu^-$ .

Pf. Since  $|\mu(E)| \leq |\mu|(E)$ ,  $\mu \ll |\mu|$ .

By the Radon-Nikodym Thm,

$\exists h \in L^1(|\mu|)$  such that

$$\mu(E) = \int_E h \, d|\mu|, \quad \forall E \in \mathcal{M}.$$

First we prove that  $|h| \leq 1$  for  $|\mu|$ -a.e.

If not, then  $\exists \varepsilon > 0$  such that

either  $h > 1 + \varepsilon$  on a set  $E$  with  $|\mu|(E) > 0$

or  $h < -(1 + \varepsilon)$  on a set  $E$  with  $|\mu|(E) > 0$ .

WLOG, suppose the first case occurs.

$$\mu(E) = \int_E h \, d|\mu|(E) \geq (1 + \varepsilon) |\mu|(E) > 0,$$

which is impossible, because  $|\mu(E)| \leq |\mu|(E)$ .



Next we show that  $|h| \geq 1$  for  $|\mu|$ -a.e.

For  $0 < r < 1$ , define

$$A_r = \{x : |h(x)| < r\}.$$

Then if  $\{E_j\}$  is a partition of  $A_r$ ,

then

$$|\mu(E_j)| = \left| \int_{E_j} h \, d|\mu| \right|$$

$$\leq \int_{E_j} |h| \, d|\mu|$$

$$\leq \int_{E_j} r \, d|\mu|$$

$$\leq r \cdot |\mu|(E_j).$$

Hence

$$\sum_{j=1}^{\infty} |\mu(E_j)| \leq r \cdot \sum_{j=1}^{\infty} |\mu|(E_j)$$

$$= r \cdot |\mu|(A_r)$$

Taking supremum over  $\{E_j\}$  gives

$$|M|(A_r) \leq r \cdot |M|(A_r)$$

$$\Rightarrow |M|(A_r) = 0.$$

So

$$|M|\left\{x: |h| < 1\right\}$$

$$= |M|\left(\bigcup_{n=1}^{\infty} A_{1/n}\right) \leq \sum_{n=1}^{\infty} |M|(A_{1/n})$$
$$= 0.$$

Hence  $|h| \geq 1$   $|M|$ -a.e.

So  $|h| = 1$   $|M|$ -a.e.

This completes the proof of ①.

Next set

$$A = \{x : h(x) = 1\}$$

$$B = \{x : h(x) = -1\}.$$

Then

$$\begin{aligned}\mu(E \cap A) &= \int_{E \cap A} h \, d|\mu| \\ &= \int_{E \cap A} 1 \, d|\mu| \\ &= |\mu|(E \cap A), \quad \textcircled{1}\end{aligned}$$

$$\begin{aligned}\mu(E \cap B) &= \int_{E \cap B} h \, d|\mu| \\ &= \int_{E \cap B} (-1) \, d|\mu| \\ &= -|\mu|(E \cap B) \quad \textcircled{2}\end{aligned}$$

From this, we can check that

$$\mu^+ = \mu|_A \quad \text{and} \quad \mu^- = -\mu|_B$$

Indeed 
$$\begin{aligned}\mu^+(E) &= \frac{1}{2}(|M|(E) + \mu(E)) \\ &= \frac{1}{2}(|M|(E \cap A) + |M|(E \cap B) + \mu(E \cap A) + \mu(E \cap B)) \\ &= \frac{1}{2}(|M|(E \cap A) + \mu(E \cap A)) \\ &= \mu(E \cap A).\end{aligned}$$

Similarly 
$$\mu^-(E) = -\mu(E \cap B).$$

③. Suppose

$$\mu = \lambda_1 - \lambda_2, \quad \text{where } \lambda_1, \lambda_2 \text{ are measures.}$$

We need to show that

$$\lambda_1 \geq \mu^+$$

$$\lambda_2 \geq \mu^-.$$

Notice by ②,  $\exists A, B$  disjoint,

such that

$$\mu^+(E) = \mu(E \cap A)$$

$$\mu^-(E) = -\mu(E \cap B).$$

So

$$\begin{aligned}\mu^+(E) &= \mu(E \cap A) \\ &= \lambda_1(E \cap A) - \lambda_2(E \cap A) \\ &\leq \lambda_1(E \cap A) \\ &\leq \lambda_1(E).\end{aligned}$$

$$\begin{aligned}\mu^-(E) &= -\mu(E \cap B) \\ &= \lambda_2(E \cap B) - \lambda_1(E \cap B) \\ &\leq \lambda_2(E \cap B) \\ &\leq \lambda_2(E).\end{aligned}$$

