Real Analysis

20-11-20

§4.4. The dual space of $L^p(\mu)$, $1<p<\infty$.

Let $1<p<\infty$. Take $q>1$ such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

Let $g\in L^q(\mu)$. Define $\Lambda_g : L^p(\mu)\to\mathbb{R}$ by

$$(*) \qquad \Lambda_g(f)=\int f g \, d\mu, \qquad \forall\ f\in L^p(\mu).$$

By the Hölder inequality

$$\int |fg|\, d\mu \leq \|f\|_p\cdot\|g\|_q < \infty$$

Hence Λ_g is well-defined, and it is a bounded linear functional on $L^p(\mu)$.

$$\|\Lambda_g\| = \sup_{\substack{f\in L^p(\mu)\\ \|f\|_p=1}} |\Lambda_g(f)| \leq \sup_{\substack{f\in L^p(\mu)\\ \|f\|_p=1}} \|g\|_q\cdot\|f\|_p$$

$$= \|g\|_q.$$

Define

$$L^p(\mu)' = \{ \text{bdd linear functionals on } L^p(\mu) \}.$$

Fact: $\quad \| \Lambda_g \| = \| g \|_q, \quad \forall \; g \in L^q(\mu).$

To prove the equality, we need to show $\exists \; f \in L^p(\mu)$ with $\|f\|_p \neq 0$ such that

$$|\Lambda_g(f)| = \|f\|_p \cdot \|g\|_q .$$

Now we take

$$f = |g|^{q-2} \cdot g = \begin{cases} 0 & \text{if } g(x) = 0 \\ |g|^{q-2} g & \text{if } g(x) \neq 0 \end{cases}$$

Notice that

$$\int |f|^p \, d\mu = \int |g|^{(q-1)p} \, d\mu = \int |g|^q \, d\mu < \infty$$

Hence $f \in L^p$,

$$\|f\|_p = \left(\int |g|^q d\mu \right)^{1/p} = \|g\|_q^{q/p}.$$

$$\Lambda_g(f) = \int g f \, d\mu$$

$$= \int g \cdot |g|^{q-2} \cdot g \, d\mu = \int |g|^q \, d\mu$$

$$= \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1}$$

$$= \|g\|_q \cdot \|g\|_q^{q/p} \qquad (q-1 = q/p)$$

$$= \|g\|_q \cdot \|f\|_p$$

Hence $\|\Lambda_g\| \geq \|g\|_q$ and so

$$\|\Lambda_g\| = \|g\|_q.$$

Remark: Let $\Phi : L^q(\mu) \to L^p(\mu)'$ be given by

$$\Phi(g) = \Lambda_g, \qquad g \in L^q(\mu)$$

Then Φ is norm preserving. So

Φ is injective. (If $\Phi(x) = \Phi(y)$,

then $\Phi(x-y) = 0 \Rightarrow 0 = \|\Phi(x-y)\|$

$$= \|x-y\|_q$$

$$\Rightarrow x = y.)$$

Thm 4.17. Let $1 < p < \infty$.

Let $\Lambda \in L^p(\mu)'$. Then $\exists$ a unique $g \in L^q(\mu)$ such that

$$\Lambda = \Lambda_g.$$

(Hence $L^p(\mu)' \approx L^q(\mu)$)

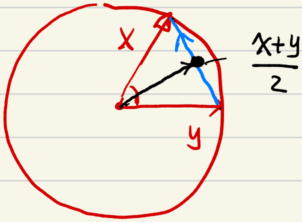
The proof of the above theorem depends on the uniform convexity of $L^p(\mu)$.

Def (Unif. Convexity).

A normed vector space X is said to be uniform convex if $\forall\, 0 < \varepsilon < 1$, $\exists\ \theta > 0$ such that for $x, y \in X$

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| < 1 - \theta.$$

Example: $\mathbb{R}^2$ with the standard norm is unif convex.



Example: Any Hilbert space (including $\mathbb{R}^n$ on $\mathbb{R}$ with the standard norm)

is unif. convex.

Because $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

(Check $\langle x+y, x+y\rangle + \langle x-y, x-y\rangle$
$= 2(\langle x,x\rangle + \langle y,y\rangle)$)

Thm 4.18. $L^p(\mu)$, $1<p<\infty$, is unif. convex.

Prop 4.19 (Clarkson's inequality).

Let $1<p<\infty$, $f, g \in L^p(\mu)$. Then

① If $p \geq 2$, we have

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right).$$

② If $p \in (1,2)$, then

$$\|f+g\|_p^q + \|f-g\|_p^q \leq 2\left(\|f\|_p^p + \|g\|_p^p\right)^{q-1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Clarkson inequality $\Rightarrow$ $L^p(\mu)$ is unif conv.

- $f, g \in L^p(\mu)$, $\|f\|_p = \|g\|_p = 1$.

If $p \geq 2$,

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{1}{2}\cdot 2 = 1$$

So if $\|f-g\|_p > \varepsilon$, then

$$\left\|\frac{f+g}{2}\right\|_p \leq \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} < 1-\theta$$

for some $\theta > 0$.

If $p \in (1,2)$, then

$$\|f+g\|_p^q + \|f-g\|_p^q \le 2 \cdot (2)^{q-1} = 2^q$$

If $\|f-g\|_p > \varepsilon$, then

$$\|f+g\|_p \le \left(2^q - \varepsilon^q\right)^{1/q}$$

$$< 2(1-\theta)$$

for some $\theta > 0$.

$\boxed{}$

$\boxed{\text{Proof of Clarkson's inequality in the case } p \ge 2:}$

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \le \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right),$$

$$\forall\, f, g \in L^p(\mu).$$

It is enough to prove

$\forall\ x, y \in \mathbb{R}$,

$$\left|\frac{x+y}{2}\right|^p + \left|\frac{x-y}{2}\right|^p \leq \frac{1}{2}\left(|x|^p + |y|^p\right)$$

Clearly the above inequality holds if $x = y = 0$.
or one of them is zero

To show the general case when $x, y \neq 0$ and $|x| \geq |y|$,
dividing $|x|^p$ to the both sides of the inequality
and letting $z = \frac{|y|}{|x|}$, we have

$$\left(\frac{1+z}{2}\right)^p + \left(\frac{1-z}{2}\right)^p \leq \frac{1}{2}\left(1 + z^p\right), \quad \text{for } 0 < z \leq 1$$

To prove the above inequality, let

$$g(z) = \left(\frac{1+z}{2}\right)^p + \left(\frac{1-z}{2}\right)^p - \frac{1}{2}\left(1 + z^p\right).$$

Then $g(0) = 2 \cdot \left(\frac{1}{2}\right)^p - \frac{1}{2} \leq 0$

$g(1) = 0$

So to show that $g(z) \leq 0$ on $(0, 1]$,
it is enough to show that $g'(z) \geq 0$ on $(0, 1]$

Taking derivative to g gives

$$g'(z) = p\left(\frac{1+z}{2}\right)^{p-1} \cdot \frac{1}{2} - p\left(\frac{1-z}{2}\right)^{p-1} \frac{1}{2} - \frac{1}{2} p z^{p-1}$$

$$= \frac{p}{2}\left[\left(\frac{1+z}{2}\right)^{p-1} - \left(\frac{1-z}{2}\right)^{p-1} - z^{p-1}\right]$$

Notice that $\frac{1+z}{2} = \frac{1-z}{2} + z$

But $(x+y)^{p-1} \geq x^{p-1} + y^{p-1}$ (since $p-1 \geq 1$)

for all $x, y \geq 0$.

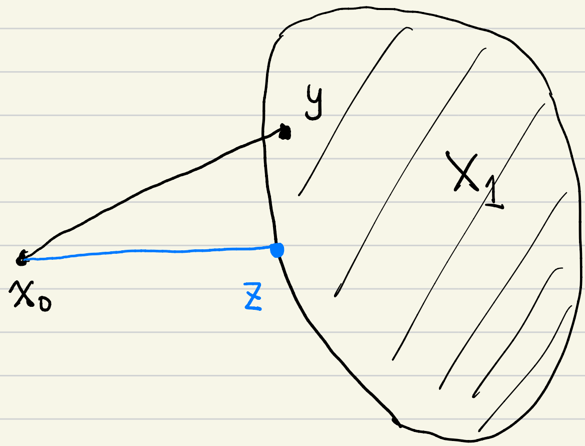
(equivalent to $(1+z)^{p-1} \geq 1 + z^{p-1}$)

Hence $g'(z) \geq 0$ on $[0,1]$ and we are done.

Thm 2.20. Let X be a unif. convex Banach space. Let X_1 be a closed (linear) subspace of X.

Let $x_0 \in X \setminus X_1$. Then $\exists$ a unique $z \in X_1$ such that

$$\|x_0 - z\| = \inf\{\|x_0 - y\| : y \in X_1\}$$



Proof: Write

$$d = \inf\{\|x_0 - y\| : y \in X_1\}.$$

We can choose $(y_n)_{n=1}^{\infty} \subset X_1$ such that

$$\lim_{n\to\infty} \| y_n - x_0 \| = d.$$

We claim that $d \neq 0$. Otherwise $y_n \to x_0$ as $n \to \infty$. But since $(y_n) \subset X_1$ and X_1 is closed, so $x_0 \in X_1$, which leads to a contraction as $x_0 \notin X_1$.

Notice that

$$\limsup_{n,m\to\infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \right\| \leqslant 2$$

and

$$\liminf_{n,m\to\infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \right\|$$

$$= \liminf_{n,m\to\infty} \left\| \frac{y_n - x_0}{d} + \frac{y_m - x_0}{d} \right\|$$

$$= \lim_{n,m\to\infty} \frac{2}{d} \left\| \frac{y_n + y_m}{2} - x_0 \right\| \qquad \left(\text{notice } \frac{y_n + y_m}{2} \in X_1 \right)$$

$$\geqslant \frac{2}{d} \cdot d = 2$$

Hence we obtain

$$\lim_{n,m\to\infty} \left\| \frac{y_n - x_0}{2\|y_n - x_0\|} + \frac{y_m - x_0}{2\|y_m - x_0\|} \right\| = 1.$$

By the unif. convexty of X, we have

$$\lim_{n,m\to\infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} - \frac{y_m - x_0}{\|y_m - x_0\|} \right\| = 0$$

which implies

$$\lim_{n,m\to\infty} \left\| \frac{y_n - x_0}{d} - \frac{y_m - x_0}{d} \right\| = 0,$$

i.e.

$$\lim_{n,m\to\infty} \| y_n - y_m \| = 0$$

So $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in X.

Hence $\exists$ $z \in X$ such that $y_n \to z$ as $n \to \infty$

Since $(y_n) \subset X_1$ and X_1 is closed, $z \in X_1$.

Now

$$\| z - x_0 \| = \lim_{n\to\infty} \| y_n - x_0 \| = d.$$

Suppose $\exists$ another $z' \in X$, so that

$$\| z' - x_0 \| = d.$$

Notice that $\| z - z' \| = \varepsilon > 0$. By the unif convexity of X

$$\left\| \frac{\frac{z - x_0}{d} + \frac{z' - x_0}{d}}{2} \right\| < 1$$

$$\Rightarrow \left\| \frac{z + z'}{2} - x_0 \right\| < d$$

which is impossible, since $\frac{z+z'}{2} \in X$, so

$$\left\| \frac{z+z'}{2} - x_0 \right\| \geq d.$$

$\square$

Lemma A: Let $\Lambda : X \to \mathbb{R}$ be a linear functional on a vector space X.

Suppose $\Lambda(x_0) \neq 0$ for some $x_0 \in X$.

Then $\forall\, x \in X$,

$$x - \frac{\Lambda(x)}{\Lambda(x_0)} \cdot x_0 \in \ker \Lambda,$$

where $\ker \Lambda = \{ y \in X : \Lambda y = 0 \}$.

pf. $\Lambda\left(x - \frac{\Lambda(x)}{\Lambda(x_0)} x_0 \right)$

$$= \Lambda x - \frac{\Lambda(x)}{\Lambda(x_0)} \cdot \Lambda(x_0)$$

$$= 0$$

$\blacksquare$

Lemma B. Let $\Lambda, \Lambda_1 : X \to \mathbb{R}$

be linearly functionals on a vector

space X. Suppose $\Lambda \neq 0$, and

$\ker(\Lambda_1) \supset \ker \Lambda$.

Then $\exists$ $c \in \mathbb{R}$ such that

$$\Lambda_1 = c\Lambda.$$

Pf. Since $\Lambda \neq 0$, $\exists$ $x_0 \in X$ such that

$$\Lambda(x_0) \neq 0.$$

By Lemma A, any vector $x \in X$

is a linear combination of x_0 and

an element in $\ker \Lambda$.

Let $c = \dfrac{\Lambda_1(x_0)}{\Lambda(x_0)}$.

set $g = \Lambda_1 - c\Lambda$.

Then $g(x_0) = \Lambda_1(x_0) - c\Lambda(x_0)$

$$= \Lambda_1(x_0) - \frac{\Lambda_1(x_0)}{\Lambda(x_0)} \cdot \Lambda(x_0)$$

$$= 0$$

and $g(y) = 0$ for $y \in \ker \Lambda$

($y \in \ker \Lambda \Rightarrow y \in \ker(\Lambda_1)$

Hence

$$g(y) = \Lambda_1(y) - c\Lambda(y) = 0 \quad)$$

But $x = \dfrac{\Lambda(x)}{\Lambda(x_0)} \cdot x_0 + y$ for some $y \in \ker(\Lambda)$

Hence

$$g(x) = \frac{\Lambda(x)}{\Lambda(x_0)} g(x_0) + g(y)$$

$$= 0$$

$\Rightarrow g = 0 \Rightarrow \Lambda_1 = c\Lambda.$ $\square$

Thm 4.17 Let $1<p<\infty$. Let $\Lambda \in L^p(\mu)'$, then $\exists\ g \in L^q(\mu)$ such that $\Lambda = \Lambda_g$.

Pf of Thm 4.17:

If $\Lambda \in L^p(\mu)'$ with $\Lambda = 0$.

Then we can take $g=0$.

Next assume $\Lambda \in L^p(\mu)'$, $\Lambda \neq 0$.

Hence $\exists\ f_1 \in L^p(\mu)$ such that

$$\Lambda(f_1) \neq 0.$$

Notice that

$$\ker \Lambda = \{ f \in L^p(\mu) : \Lambda f = 0 \}$$

is a closed linear subspace of $L^p(\mu)$.

Since $L^p(\mu)$ is unif. convex, by Thm 4.19,

$\exists$ a unique $h_0 \in \ker \Lambda$ such that

$$\| h_0 - f_1 \|_p = \inf \{ \| f - f_1 \|_p : f \in \ker(\Lambda) \}$$

This implies that for any $f \in \ker(\Lambda)$,

$$\varphi(t) = \| h_0 + tf - f_1 \|_p^p$$
$$= \int | h_0 + tf - f_1 |^p \, d\mu$$

gets minimized at $t = 0$.

Claim: $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable over $\mathbb{R}$ with

$$\varphi'(t) = \int | h_0 + tf - f_1 |^{p-2} (h_0 + tf - f_1) \cdot f \, d\mu .$$

Exer. Prove the above claim.

Hint: ① $\dfrac{d(|a+bt|^p)}{dt} = |a+bt|^{p-2}(a+bt) \cdot b$

$\forall\, a, b \in \mathbb{R}$, $\forall\, t \in \mathbb{R}$.

② Use the Dominated Convergence Thm.

Now since φ is diff on $\mathbb{R}$

and φ takes the minimum at $t=0$,

This implies

$$\varphi'(0) = 0$$

But $\varphi'(0) = \int |h_0 - f_1|^{p-2}(h_0 - f_1) \cdot f \, d\mu$

$$= \int g f \, d\mu = 0.$$

Next we define

$$g = |h_0 - f_1|^{p-2} \cdot (h_0 - f_1).$$

Then

$$\int |g|^q \, d\mu = \int |h_0 - f_1|^{(p-1)q} \, d\mu$$

$$= \int |h_0 - f_1|^p \, d\mu < \infty$$

So $g \in L^p(\mu)$.

Since $h_0 \neq f_1$ $\left(\| h_0 - f_1 \|_p > 0 \right)$

we have $\| g \|_q \neq 0$.

Recall that

$$\int g f \, d\mu = 0 \quad \forall f \in \ker \Lambda.$$

Hence

$$\ker \Lambda \subset \ker \Lambda_g$$

Since $\Lambda \neq 0$, by Lemma B,

$\exists$ c such that

$$\Lambda_g = c\Lambda$$

Hence $\Lambda = \frac{1}{c}\Lambda_g = \Lambda_{\frac{1}{c}g}$.

$\square$