Real Analysis 20-10-23.

Chap 3 Lebesgue measures and Hausdorff measures.

§ 3.1 Lebesgue measures on $\mathbb{R}^n$.

Let $R = \prod_i [a_i, b_i]$ be a closed rectangles in $\mathbb{R}^n$.

Let $|R|$ denote the volume of R, i.e. $|R| = \prod_i (b_i - a_i)$.

A cube in $\mathbb{R}^n$ is a rectangle R such that $b_j - a_j$ are the same.

Def. The n-dim Lebesgue measure on $\mathbb{R}^n$ is defined by

$$\mathcal{L}^n(E) = \inf \left\{ \sum_{i=1}^{\infty} |R_i| : \bigcup_i R_i \supset E, \ R_i \text{ closed cubes} \right\}$$

- $\mathcal{L}^n$ is generated by the gauge $(\mathcal{R}, |\cdot|)$, where $\mathcal{R}$ is the collection of closed cubes in $\mathbb{R}^n$, $|R|$ is the volume of R.

Facts and remarks:

① Equivalently

$$\mathcal{L}^n(E) = \inf \left\{ \sum_{i=1}^{\infty} |R_i| : E \subset \bigcup_i R_i , \ R_i \text{ open cubes} \right\}.$$

② $\forall \delta > 0$,

$$\mathcal{L}^n(E) = \inf \left\{ \sum_{i=1}^{\infty} |R_i| : E \subset \bigcup_i R_i , \ R_i \text{ closed cubes of diameter} \leq \delta \right\}$$

③ $\mathcal{L}^n$ is a metric outer measure, so it is a Borel measure.

④ $\mathcal{L}^n(R) = |R|$ for each cube R.

⑤ Every set in $\mathbb{R}^n$ is outer regular (w.r.t. $\mathcal{L}^n$)

⑥ Every measure set is inner regular.

⑦ The σ-algebra of measurable sets in the completion of the Borel σ-algebra.

⑧ $\mathcal{L}^n$ is translation invariant,

i.e. $\mathcal{L}^n(E+x) = \mathcal{L}^n(E)$, $\forall E \subset \mathbb{R}^n$, $\forall x \in \mathbb{R}^n$.

⑨ For any Borel measure μ on $\mathbb{R}^n$, if μ is translation invariant on $\mathbb{R}^n$, then $\mu = c \cdot \mathcal{L}^n$ for some c.

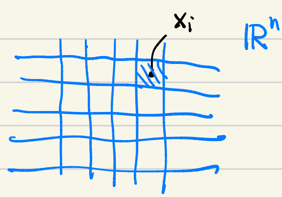
Alternatively, Lebesgue measure can be constructed using Riesz representation Thm.

To see this, we define an operator L on $C_c(\mathbb{R}^n)$ by

$$L(f) = \int f \, dx, \qquad f \in C_c(\mathbb{R}^n)$$

$\uparrow$

(Riemann integral via Darboux sum)



$$\sum_i f(x_i) \cdot (\Delta x_i)$$

- $L(f)$ is well-defined (any $f \in C_c(\mathbb{R}^n)$ is Riemann integrable)
- L is linear
- L is positive.

Hence L is a positive linear functional over $C_c(\mathbb{R}^n)$.

By Riesz representation Thm, $\exists$ a Borel measure μ_R on $\mathbb{R}^n$ such that μ_R is finite on compact sets and

$$L(f) = \underbrace{\int f \, d\mu_R}_{\text{Leb. integration}}$$

Prop 3.1 $\quad \mu_R = \mathcal{L}^n$.

Pf. We first prove that μ_R is translation invariant.

Let G be open in $\mathbb{R}^n$, let $f \in C_c(\mathbb{R}^n)$ such that $f < G$.

Let $x_0 \in \mathbb{R}^n$, define $g(x) = f(x+x_0)$. It is direct to see that

$$g < G - x_0.$$

Moreover (using the def of R.I.)

$$\int_G f \, dx = \int_{G-x_0} g \, dx$$

Hence $L(f) = L(g)$, so

$$\mu_R(G) = \mu_R(G - x_0). \quad \left(\text{using } \mu_R(G) = \sup\{ L(f) : f < G \} \right)$$

Using the outer regularity of μ_R, we see that

$$\mu_R(E) = \mu_R(E - x_0) \quad \text{for any } E \subset \mathbb{R}^n.$$

Hence μ_R is translation invariant.

Therefore

$$\mu_R = c \cdot \mathcal{L}^n \quad \text{for some } c \geq 0.$$

Next we prove $c = 1$.

Let $R_0 = (0,1)^n$, let $R_\varepsilon = (\varepsilon, 1-\varepsilon)^n$ for $0 < \varepsilon < \frac{1}{2}$

For $\overline{R_\varepsilon} < f < R_0$,

$$\mu_R(R_0) \geq L(f) = \int f \, dx \geq |\overline{R_\varepsilon}| = (1 - 2\varepsilon)^n$$

$\left(\text{because } f \equiv 1 \text{ on } \overline{R_\varepsilon} \text{ and } f \geq 0 \right)$

Letting $\varepsilon \to 0$ gives $\mu_R(R_0) \geq 1$.

However for any $f < R_0$,

$$\int f\,dx = \int_{R_0} f\,dx \le |R_0| \quad \left(f \text{ is supported on } R_0 \text{ and } f \le 1 \right)$$

Hence

$$\mu_R(R_0) = \sup\{ L(f) : f < R_0 \} \le |R_0| = 1$$

Therefore $\mu_R(R_0) = 1 = \mathcal{L}^n(R_0) \Rightarrow c = 1$.

$\boxed{/\!/\!/}$.

Prop. 3.2 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then $\exists$ a constant $\Delta(T) \in \mathbb{R}$ such that

$$\mathcal{L}^n(TE) = \Delta(T)\,\mathcal{L}^n(E), \quad \forall E \subset \mathbb{R}^n.$$

Furthermore $\Delta(T) = 1$ if T is a rotation or a reflection.

Pf. Define

$$\mu(E) = \mathcal{L}^n(TE), \quad E \subset \mathbb{R}^n.$$

Clearly μ is an outer measure on $\mathbb{R}^n$.

Next we assume T is non-singular. Notice that for $x, y \in \mathbb{R}^n$,

$$x - y = T^{-1}(Tx - Ty)$$

Hence

$$\| x - y \| \leq \| T^{-1} \| \cdot \| Tx - Ty \|$$

$\left(\| \vec{a} \| = \sqrt{a_1^2 + \cdots + a_n^2} \right.$ for $\left. \vec{a} = (a_1, \cdots, a_n) \right)$

It implies $\| Tx - Ty \| \geq \frac{1}{\| T^{-1} \|} \cdot \| x - y \|$

Hence for $A, B \subset \mathbb{R}^n$

$$d(A, B) > 0 \Rightarrow d(TA, TB) > 0$$

It implies that μ is a metric outer measure.

(Recall $\mu(E) = \mathcal{L}^n(TE)$)

Therefore $\mu = C \cdot \mathcal{L}^n$ for some constant.

Next assume T is singular, then $T(\mathbb{R}^n)$ is contained in a hyplane of $\mathbb{R}^n$ so,

$\mathcal{L}^n(T(\mathbb{R}^n)) = 0$. Then $\mu = 0$

In the case that T is a rotation or a reflection, then

$$B = T(B) \quad \text{where } B \text{ is the unit ball centered}$$

at the origin.

$$\text{So} \quad \mu(B) = \mathcal{L}^n(TB) = \mathcal{L}^n(B) \Rightarrow c = 1.$$

$\blacksquare$

§ 3.1 Lebesgue measure on $\mathbb{R}$.

- Non-measurable subsets of $\mathbb{R}$.

Q: Is every subset of $\mathbb{R}$ measurable wrt $\mathcal{L}^1$?

No. Answered by Vitali at 1905.

Vitali's construction of non-measurable sets.

We introduce a relation "$\sim$" on $\mathbb{R}$ by

$$x \sim y \quad \text{if and only} \quad x-y \in \mathbb{Q}$$

(the set of rationals)

One can check that $\sim$ is an equivalence relation

- $x \sim y \Rightarrow y \sim x$
- $x \sim y,\ y \sim z \Rightarrow x \sim z$
- $x \sim x$

Then

$$\mathbb{R} = \bigsqcup_{\alpha} \mathcal{E}_\alpha$$

where $\mathcal{E}_\alpha$ are equivalence classes.

By the axiom of choice, by picking x_α from $\mathcal{E}_\alpha$ for each α,

we form a set $\{x_\alpha\}_\alpha =: \mathcal{E}$

We call $\mathcal{E}$ a Vitali set. Then as a remarkable fact, we have

$$\mathbb{R} = \bigsqcup_{q \in \mathbb{Q}} (\mathcal{E} + q)$$

where the union is disjoint. (check it).

<u>Prop 3</u>: Every set in $\mathbb{R}$ with positive Lebesgue measure contains a non-measurable subset.

Pf. It is equivalent to prove the following:

If $A \subset \mathbb{R}$ such that all subsets of A are measurable, then $\mathcal{L}^1(A) = 0$.

Fix such a set A. For $q \in \mathbb{Q}$, we define

$$A_q = A \cap (\mathcal{E} + q),$$

where $\mathcal{E}$ is a Vitali set. Then

$$A = \bigsqcup_{q \in \mathbb{Q}} A_q \qquad \Big(\text{because } \mathbb{R} = \bigsqcup_{q \in \mathbb{Q}} (\mathcal{E} + q)\Big)$$

Since A_q are measurable by assumption, we have

$$\mathcal{L}^1(A) = \sum_{q \in \mathbb{Q}} \mathcal{L}^1(A_q)$$

Now we need to show that $\mathcal{L}^1(A_q) = 0$ for each $q \in \mathbb{Q}$.

By the inner regularity of $\mathcal{L}^1$, it is enough to show that

$\mathcal{L}^1(K) = 0$ for each compact subset K of A_q.

Fix such a compact $K \subset A_q$. Define

$$H = \bigcup_{\substack{r \in \mathbb{Q} \\ |r| \leq 1}} K + r$$

For rationals $r_1 \neq r_2$, since $K + r_1 \subset \mathcal{E} + q + r_1$
$K + r_2 \subset \mathcal{E} + q + r_2$ (they are disjoint)

Hence the union in defining H is disjoint

So $\mathcal{L}^1(H) = \sum_{\substack{r \in \mathbb{Q} \\ |r| \leq 1}} \mathcal{L}^1(K + r) = \sum_{\substack{r \in \mathbb{Q} \\ |r| \leq 1}} \mathcal{L}^1(K)$.

If $\mathcal{L}^1(K) > 0 \Rightarrow \mathcal{L}^1(H) = +\infty$

however this is impossible because H is a bounded set

Hence $\mathcal{L}^1(K) = 0$. $\blacksquare$.

Remark: Any Vitali set $\mathcal{E}$ is non-measurable.

pf. Suppose on the contrary $\mathcal{E}$ is measurable.

$$\mathbb{R} = \underbrace{\bigsqcup_{q \in \mathbb{Q}}}_{\text{(disjoint)}} \underbrace{(\mathcal{E} + q)}_{\text{measurable}}$$

Hence $\displaystyle \mathcal{L}^1(\mathbb{R}) = \sum_{q \in \mathbb{Q}} \mathcal{L}(\mathcal{E} + q)$

$$= \sum_{q \in \mathbb{Q}} \mathcal{L}(\mathcal{E})$$

which implies $\mathcal{L}(\mathcal{E}) > 0$.

However for any compact subset K of $\mathcal{E}$,

Using the same trick,

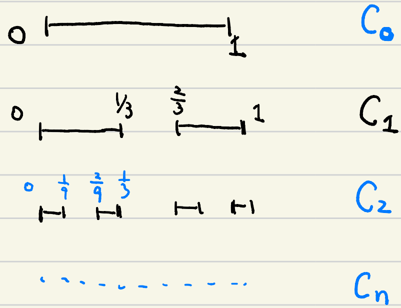
$$H = \bigsqcup_{\substack{r \in \mathbb{Q} \\ |r| \leq 1}} K + r$$

H is bdd, $\Rightarrow \mathcal{L}^1(K) = 0$

By the inner regularity of $\mathcal{E}$ $\Rightarrow \mathcal{L}^1(\mathcal{E}) = 0$.

- Cantor set and Cantor function.

$\boxed{\text{Middle-third Cantor set } C.}$ The construction is the following:



$$C_0 \supset C_1 \supset C_2 \cdots \supset C_n \supset \cdots$$

Let $C = \bigcap_{n=0}^{\infty} C_n$

We call C the middle-third Cantor set.

Below we list some properties of C

- C is compact, non-empty, uncountable.
- nowhere dense
- perfect (no isolated points)
- $\mathcal{L}^1(C) = 0$
- $C = \left\{ x = \sum_{n=1}^{\infty} a_n 3^{-n} : \quad a_n = 0 \text{ or } 2 \right\}$

Next we introduce the $\boxed{\text{Cantor function } F}$

Define $F : C \to [0,1]$ by

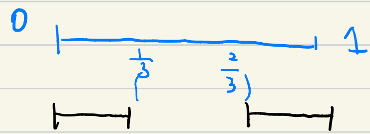
$$x = \sum_{n=1}^{\infty} a_n 3^{-n} \longmapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot 2^{-n}$$

- Then the function F is continuous, increasing, surjective.

- Moreover if (a,b) is an interval in the complement of C_n, then

$$F(a) = F(b)$$

For instance $a = \frac{1}{3}$, $b = \frac{2}{3}$



Recall that $\frac{1}{3} = 0.0222\cdots2\cdots$ (in base 3)

$\frac{2}{3} = 0.20000\cdots$ (in base 3)

$$F\left(\tfrac{1}{3}\right) = 0.0111111\cdots \quad \text{(in base 2)}$$

$$F\left(\tfrac{2}{3}\right) = 0.1000000 \quad \text{(in base 2)}$$

Now we extend $F: C \to [0,1]$

to $F: [0,1] \to [0,1]$

such that F is constant in each component of $[0,1] \setminus C$

We call $F: [0,1] \to [0,1]$ the Cantor function.

Then F satisfies the following properties:

① F is increasing on $[0,1]$

② F is cts

③ $F'(x) = 0$ for $\mathcal{L}^1$-a.e $x \in [0,1]$

④ $F(C) = [0,1]$

⑤ $|F(x) - F(y)| \leq D\, |x-y|^{\frac{\log 2}{\log 3}}$ $\forall\, x, y \in [0,1]$

and D is a constant.

Next we build an invertible cts function $h : [0,1] \to [0,2]$

such that h^{-1} maps some non-measurable sets into measurable subset.

Let $h(x) = x + F(x)$ for $x \in [0,1]$

Notice that h is cts and strictly increasing.

$h(0) = 0 + F(0) = 0$, $h(1) = 1 + F(1) = 2$.

So $\boxed{h \text{ is a homeomorphism}}$ from $[0,1]$ to $[0,2]$.

Key point : $\mathcal{L}^1(h(C)) = 1$.

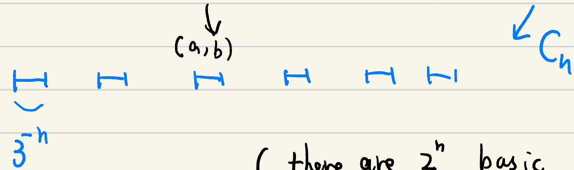
Notice that $C = \bigcap_{n=1}^{\infty} C_n$, $C_n \downarrow$

So $h(C) = \bigcap_{n=1}^{\infty} h(C_n)$, $h(C_n) \downarrow$

By the continuity of Leb. measure

$$\mathcal{L}^1(h(C)) = \lim_{n\to\infty} \mathcal{L}^1(h(C_n)).$$

Notice that for each basic interval in C_n



(there are 2^n basic intervals in C_n)

$$\mathcal{L}^1(h(a,b)) = 2^{-n} + 3^{-n}$$

$$h(x) = x + F(x) \qquad (b-a = 3^{-n})$$

$$\begin{aligned}\mathcal{L}^1(h(a,b)) &= b + F(b) - a - F(a) \\ &= (b-a) + (F(b) - F(a)) \\ &= 3^{-n} + 2^{-n}\end{aligned}$$

(using the 3-adic expansions of b, a.)

I leave it as an exer.

Hence
$$\mathcal{L}^1(h(C_n)) = 2^n(2^{-n} + 3^{-n}) = 1 + \left(\frac{2}{3}\right)^n.$$

Letting $n \to \infty$ gives $\mathcal{L}^1(h(C)) = 1$.

Now by Prop 3.3, $\exists$ a non-measurable $A \subset h(C)$

Let $B = h^{-1}(A) \subset C$.

So $h(B) = A$, with A being non-measurable.

However $B \subset C$, so $\mathcal{L}^1(B) = 0 \Rightarrow B$ is measurable.

• Remark: If $f : [a,b] \to [c,d]$ is a homeomorphism, then f maps every Borel subset of $[a,b]$ into a Borel subset of $[c,d]$.

(Exer.)