Real Analysis

20-09-11.

Chapter 1. Integration on measure spaces.

§1.1 Measurable spaces and measurable functions.

Let $X \neq \emptyset$. Let $\mathcal{P}_X$ be the power set of X, i.e.

$$\mathcal{P}_X = \{ Y : Y \subset X \}.$$

Def. (σ-algebra) $\mathcal{M} \subset \mathcal{P}_X$ is called a σ-algebra on X if

(i) $X \in \mathcal{M}$;

(ii) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ $(A^c := X \setminus A)$

(iii) If $A_k \in \mathcal{M}$, $k \geq 1$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$.

Remark: (i)-(iii) imply that

- $\emptyset \in \mathcal{M}$.
- If $A_k \in \mathcal{M}$, $k \geq 1$, then $\bigcap_{k=1}^{\infty} A_k \in \mathcal{M}$.

  (using $\bigcap_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty} A_k^c \right)^c$)

Hence a σ-algebra is closed under countable union, intersection and complement.

Example: $\cdot\ \mathcal{P}_X$

$\cdot\ \{\emptyset, X\}$

Example: Let $S \subset \mathcal{P}_X$.

Define

$$\mathcal{M}(S) = \bigcap \text{ all } \sigma\text{-algebras on } X \text{ containing } S.$$

We call $\mathcal{M}(S)$ the smallest σ-algebra containing S.

(or the σ-algebra generated by S).

Example: Let X be a topological space.

Let $\mathcal{B}_X$ be the σ-algebra on X generated by the class of open sets in X.

We call $\mathcal{B}_X$ the Borel σ-algebra on X.

Def. A pair $(X, \mathcal{M})$ is said to be a measurable space (可测空间) if $\mathcal{M}$ is a σ-algebra on X.

Def: A function $f: X \to \mathbb{R}$ is said to be measurable if $f^{-1}(G) \in \mathcal{M}$, $\forall$ open $G \subset \mathbb{R}$.

Remark: Equivalently, f is measurable if

$$f^{-1}(a,b) \in \mathcal{M}, \quad \forall\ a, b \in \mathbb{R}, \quad a<b.$$

(Using the fact that every open set in $\mathbb{R}$ is the countable union of finite open intervals)

Prop 1.1. $f: X \to \mathbb{R}$ is measurable iff one of following properties holds:

① $f^{-1}(a,b) \in \mathcal{M}, \quad \forall\ a, b \in \mathbb{R}, \quad a<b.$

② $f^{-1}(a,\infty) \in \mathcal{M}, \quad \forall\ a \in \mathbb{R}.$

③ $f^{-1}[a,\infty) \in \mathcal{M}, \quad \forall\ a \in \mathbb{R}$

④ $f^{-1}(-\infty,a) \in \mathcal{M}, \quad \forall\ a \in \mathbb{R}$

⑤ $f^{-1}(-\infty,a] \in \mathcal{M}, \quad \forall\ a \in \mathbb{R}.$

Pf. WLOG, we prove

f is measurable $\Leftrightarrow$ ② holds.

Clearly, "$\Rightarrow$" holds.

Now to prove "$\Leftarrow$", suppose ② holds.

We want to prove ① holds.

Notice that

$$f^{-1}[a,\infty) = \bigcap_{n=1}^{\infty} f^{-1}\left(a-\tfrac{1}{n},\infty\right) \in \mathcal{M}.$$

So

$$f^{-1}(a,b) = f^{-1}(a,\infty) \setminus f^{-1}[b,\infty) \in \mathcal{M} \quad \blacksquare.$$

Prop 2. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be continuous.

Let $f : X \to \mathbb{R}$ be measurable.

Then $\Phi \circ f : X \to \mathbb{R}$ is measurable.

Pf. $\forall$ open G in $\mathbb{R}$, by continuity,

$\Phi^{-1}(G)$ is open in $\mathbb{R}$.

So $(\Phi \circ f)^{-1}(G) = f^{-1}\left(\Phi^{-1}(G)\right) \in \mathcal{M}$. $\blacksquare$.

Remark: The assumption $\Phi : \mathbb{R} \to \mathbb{R}$ can be relaxed to

$\Phi : V \to \mathbb{R}$ where V is open and $V \supset \operatorname{range}(f)$.

Prop 1.3 (i) All measurable functions on X form a vector space.

(ii) If f is measurable, then so are

$$f^2, \ |f|, \ f^+, \ f^-.$$

$$\left(f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases} \right.$$

$$\left. f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise.} \end{cases} \right)$$

(iii) If f, g are measurable then $f \cdot g$ is measurable.

(iv) If f, g are measurable and $g \neq 0$ then f/g is measurable.

Pf. (i) It suffices to prove that if f, g are measurable, then so is $f+g$.

$$(f+g)^{-1}(a, \infty) = \bigcup_{\substack{s,t \in \mathbb{Q} \\ s+t > a}} \left(f^{-1}(s,\infty) \cap g^{-1}(t,\infty) \right)$$

$$\in \mathcal{M}.$$

(ii). Let $\Phi(x) = x^2$. By Prop 1.2, $\Phi \circ f$ is measurable. But $\Phi \circ f = f^2$, and we are done.

(iii) $$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2 \right).$$

(iv) Since $g \neq 0$, $\text{range}(g) \subset \mathbb{R} \setminus \{0\}$.

Set $\Phi : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ by $\Phi(x) = \frac{1}{x}$, which is cts

So by Prop 1.2, $\Phi \circ g = \frac{1}{g}$ is measurable.

Then by (iii), $\frac{f}{g} = f \cdot \frac{1}{g}$ is measurable. ☒

Prop 1.4. Let f_k, $k \geq 1$, be measurable.

Then the following are also measurable:

$$\sup_{k\geq 1} f_k, \quad \inf_{k\geq 1} f_k, \quad \overline{\lim_{k\to\infty}} f_k, \quad \underline{\lim_{k\to\infty}} f_k.$$

Pf. Let $g = \sup_{k \geq 1} f_k$.

$$g^{-1}(a,\infty) = \bigcup_{k=1}^{\infty} f_k^{-1}(a,\infty) \in \mathcal{M}.$$

$\left(g(x) > a \iff \exists\, k \text{ such that } f_k(x) > a \right).$

Hence g is measurable.

The proof for the measurability of $\inf_k f_k$ is similar.

$$\overline{\lim_{k\to\infty}} f_k = \inf_{k\geq 1} \left(\sup_{j\geq k} f_j \right) \quad \checkmark$$

$$\underline{\lim_{k\to\infty}} f_k = \sup_{k\geq 1} \left(\inf_{j\geq k} f_j \right) \quad \checkmark$$

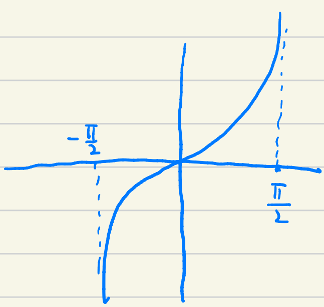
$\blacksquare$.

§1.2 Extended real numbers.

set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, and call it the extended real number system.

$\overline{\mathbb{R}}$ can be viewed as the image of the function

$$\phi(x) = \tan x, \qquad x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$



set $\tan(-\frac{\pi}{2}) = -\infty$, $\tan(\frac{\pi}{2}) = +\infty$.

Topological structure of $\overline{\mathbb{R}}$:

$E \subset \overline{\mathbb{R}}$ is open iff $\phi^{-1}(E)$ is open in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

By the above def, $E \subset \overline{\mathbb{R}}$ is open iff E is the countable union of intervals of the form

$[-\infty, a)$, (a, b), $(a, +\infty]$.

Def: A function $f: X \to \overline{\mathbb{R}}$ is said to be measurable if

$$f^{-1}(G) \in \mathcal{M} \text{ for any open } G \text{ in } \overline{\mathbb{R}}.$$

Facts: $f: X \to \overline{\mathbb{R}}$ is measurable

iff $f^{-1}(a, b)$, $f^{-1}(\{+\infty\})$, $f^{-1}(\{-\infty\}) \in \mathcal{M}$.

Prop 1.3, Prop 1.4 also hold for $\overline{\mathbb{R}}$-valued functions.

§1.3 Measure spaces.

Let $(X, \mathcal{M})$ be a measurable space.

Def. A measure μ on $(X, \mathcal{M})$ is a function from $\mathcal{M}$ to $[0, +\infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) (countable additivity)

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n), \text{ provided}$$

$A_n \in \mathcal{M}$ are mutually disjoint.

Simple facts.

- (finite additivity)

$$\mu\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} \mu(A_n) \text{ if } A_n \in \mathcal{M} \text{ are disjoint.}$$

- If $A \subset B$, then $\mu(B) = \mu(A) + \mu(B \setminus A)$.

  $(B = A \cup (B \setminus A))$

Hence $\mu(A) \leq \mu(B)$.

• (Countable sub-additivity)

Let $A_n \in \mathcal{M}$, $n \geq 1$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu(A_n).$$

Justification: Let $B_1 = A_1$,

$B_2 = A_2 \setminus A_1$

$\cdots$

$B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$

$\cdots$

Then $B_1, B_2, \cdots$ are mutually disjoint, $B_n \in \mathcal{M}$

$B_n \subset A_n$, moreover

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Hence
$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mu(B_n) \\ &\leqslant \sum_{n=1}^{\infty} \mu(A_n). \end{aligned}$$

Def. The triple $(X, \mathcal{M}, \mu)$ is called a measure space.

Prop 1.5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) If $A_k \in \mathcal{M}$, $k \geq 1$, is increasing in the sense

$$A_1 \subset A_2 \subset \cdots$$

then $$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

(ii) If $A_k \in \mathcal{M}$, $k \geq 1$, is decreasing in the sense

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

and assume $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

Pf. Let us first prove (i).

Write $B_1 = A_1$

$\cdots\cdots$

$B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$

$\cdots\cdots$

Then $B_1, B_2, \cdots,$ are mutually disjoint,

Moreover

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

$$\bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n \qquad \text{(check it).}$$

Hence

$$\begin{aligned}
\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) &= \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) \\
&= \sum_{n=1}^{\infty} \mu(B_n) \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n) \\
&= \lim_{k\to\infty} \mu\Big(\bigcup_{n=1}^{k} B_n\Big) \\
&= \lim_{k\to\infty} \mu\Big(\bigcup_{n=1}^{k} A_n\Big) \\
&= \lim_{k\to\infty} \mu(A_k) \qquad \text{(since } A_n \text{ is } \cancel{\text{there}} \text{ increasing)}
\end{aligned}$$

Now we prove (2).

Notice that

$$A_1 \supset A_2 \supset \cdots$$

So

$$A_1 \setminus A_1 \subset A_1 \setminus A_2 \subset \cdots \subset A_1 \setminus A_n \subset \cdots$$

Hence by (1)

$$\mu\left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n) \right) = \lim_{n\to\infty} \mu(A_1 \setminus A_n).$$

So

$$\mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right) = \lim_{n\to\infty} \mu(A_1 \setminus A_n).$$

$*$

Since $\mu(A_1) < \infty$,

So $\mu(A_1) = \mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

Hence $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right)$

Similarly

$$\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n).$$

Plugging the above two equations into (*), we obtain the desired identity.

Remark: The assumption $\mu(A_1) < \infty$ can not be dropped in Prop 1.5.

Here is a counterexample:

Let $\mu = \mathcal{L}^1$ on $\mathbb{R}$.

Let $A_n = (n, +\infty)$, $n = 1, 2, \dots$

Then $A_n \downarrow$

However $\bigcap_{n=1}^{\infty} A_n = \emptyset$

But

$$0 = \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \neq \lim_{n\to\infty} \mu(A_n) = +\infty .$$

§1.4. Integration on measure spaces.

Basically we define integration in 3 steps.

(1) Integration of non-negative simple functions

(2) Integration of non-negative measurable function,

(3) Integration of measurable functions.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Def. (simple functions).

A simple function on X is a measurable real function which only take finite many values.

It has a standard form

$$s(x) = \sum_{j=1}^{N} \alpha_j \chi_{E_j}(x),$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_N$ and

$$E_j = \{x : s(x) = \alpha_j\} \in \mathcal{M}.$$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

Remark: In general, any function

$$s(x) = \sum_{j=1}^{N} \gamma_j \chi_{E_j}(x)$$

$$(\text{where } E_j \in M)$$

is a simple function.

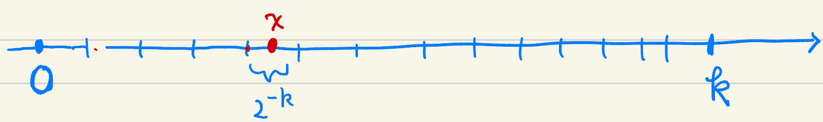
Thm 1.6. Let $f : X \to \overline{\mathbb{R}}$ be a non-negative $\overline{\mathbb{R}}$-valued measurable function. Then $\exists$ a sequence $(S_n)_{n=1}^{\infty}$ of non-negative simple functions such that

(1) $\quad S_n \leq S_{n+1}, \quad \forall\, n \geq 1,$

(2) $\quad f = \lim_{n\to\infty} S_n .$

Pf. Let us construct a sequence of functions φ_k, $k \geq 1$, from $[0,\infty] \to [0,\infty)$ by

$$\varphi_k(x) = \begin{cases} \dfrac{j}{2^k} & \text{if } x \in \left[\dfrac{j}{2^k}, \dfrac{j+1}{2^k}\right) \\ & \quad \text{for } j = 0, 1, \cdots, 2^k \cdot k - 1 \\ k & \text{otherwise} \end{cases}$$



Then $\varphi_k(x) \uparrow x$, for any $x \geq 0$.

Take $S_k(x) = \varphi_k(f(x))$, $k = 1, 2, \ldots$

It is readily checked that S_k are simple.

and $S_k(x) \nearrow f(x)$. $\blacksquare$.