Exercise 4

1. We continue our study of the Lebesgue measure beginning in Ex 3. Show that

   (a) $\mathcal{L}^n$ is a Borel measure.

   (b) For every set E, there exists a sequence of open sets $\{G_k\}$ satisfying $E \subset G_k$ and
   $$\mathcal{L}^n(E) = \lim_{k\to\infty} \mathcal{L}^n(G_k) \ .$$

   (c) For every measurable set A, there exists a sequence of compact sets $\{K_j\}$ satisfying $K_j \subset A$ and
   $$\mathcal{L}^n(A) = \lim_{j\to\infty} \mathcal{L}^n(K_j) \ .$$
   Hint: First assume A is bounded.

2. Let $(\mathbb{R}^n, \mathcal{B}, \mu)$ be a measure space where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$. Suppose that μ is translational invariant, i.e., $\mu(E+x) = \mu(E)$, $\forall x \in \mathbb{R}^n$, $E \in \mathcal{B}$, and that μ is non-trivial in the sense that $0 < \mu([0,1]^n) < \infty$. Show that μ is a constant multiple of the Lebesgue measure on $\mathbb{R}^n$ when restricted to $\mathcal{B}$.

3. Let X be a metric space and $\mathcal{C}$ be a subset of $\mathcal{P}_X$ containing the empty set and X. Assume that there is a function $\rho : \mathcal{C} \to [0,\infty]$ satisfying $\rho(\phi) = 0$. For each $\delta > 0$, show that (a)
   $$\mu_\delta(E) = \inf\Big\{ \sum_k \rho(C_k) : E \subset \bigcup_k C_k, \quad \text{diameter}(C_k) \le \delta \Big\}$$
   is an outer measure on X, and (b) $\mu(E) = \lim_{\delta\to 0} \mu_\delta(E)$ exists and is also an outer measure on X.

4. Consider in the previous problem the Euclidean space $\mathbb{R}^n, \mathcal{C} = \mathcal{P}_X$ and $s \in [0,\infty)$. Let
   $$\rho(C) = (\text{diam } (C))^s \ ,$$
   where the diameter of C is given by $\sup_{x,y\in C} |x-y|$. Show that the resulting outer measures are Borel measures.

5. Let X be a metric space and $C(X)$ the collection of all continuous real-valued functions in X. Let $\mathcal{A}$ consist of all sets of the form $f^{-1}(G)$ which $f \in C(X)$ and G is open in $\mathbb{R}$.

The "Baire σ-algebra" is the σ-algebra generated by $\mathcal{A}$. Show that the Baire σ-algebra coincides with the Borel σ-algebra $\mathcal{B}$.

6. Identify the Riesz measures corresponding to the following positive functionals ($X = \mathbb{R}$):

   (a) $\Lambda_1 f = \int_a^b f\,dx$, and

   (b) $\Lambda_2 f = f(0)$.

7. Let c be the counting measure on $\mathbb{R}$,

$$c(A) = \begin{cases} \#A, & A \neq \phi, \\ 0, & A = \phi. \end{cases}$$

   Is there a positive functional

$$\Lambda f = \int f\,dc \quad ?$$