The Chinese University of Hong Kong
MATH5011 Real Analysis I 2020-2021
Mid-term Examination

Oct 30 9:30am–12:30pm (noon)

Instructions

- This is an open-book examination. You may **ONLY** refer to printed/written materials during the examination. Assessing information on the internet is not allowed.
- You are allowed to use a calculator in the approved list during the examination.
- You shall take the examination in isolation and shall not communicate with any person during the examination other than the course teacher(s) concerned. Please kindly be reminded of the following regulations enforced by the university:

  Honesty in Academic Work: *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*

- There are a total of 100 points.
- Answer **ALL** questions.
- Show all steps clearly in your working. **NO** point will be given if suffcient justification is not provided.
- Only handwritten answers on papers or electronic devices will be accepted. Typed answer will **NOT** be accepted.
- Please follow the instruction of submission below:
  1. Write your answers on papers or electronic devices. Only handwritten answers will be accepted and typed answer will **NOT** be accepted.
  2. Scan or take photos of your work (if you write on papers).
  3. Combine your work into a single pdf file.
  4. Name the pdf file by your student id (e.g. 1155123456.pdf).
  5. You must upload the pdf file to Blackboard if you are students from CUHK, or send the pdf file to me via email (djfeng@math.cuhk.edu.hk) if your are from other universities, before 12:30pm (noon), 30 OCT, 2020. Mark deduction will be made for late submission.
  6. Please check your file carefully to make sure no missing pages.

Answer all the six questions.

1. (15 pts) Let $(X, \mathcal{M}, \lambda)$ be a measure space.

   (a) Let $A_k \in \mathcal{M},\ k \geq 1$. Show that the set
   $$E = \{x \in X : \ x \text{ belongs to exactly 2020 many } A_k\},$$
   is measurable.

   (b) Assume that $\sum_{k=1}^{\infty} \lambda(A_k) < \infty$. Show that the set
   $$F = \{x \in X : \ x \text{ belongs to infinitely many } A_k\} \ ,$$
   is a null set.

2. (15 pts)

   (a) Let f, g be two measurable functions on a measurable space $(X, \mathcal{M})$. Show that the sum $f + g$ and the product $h = fg$ are measurable.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f^{-1}(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$, where $\mathcal{B}$ stands for the Borel σ-algebra in $\mathbb{R}$.

3. (20 pts) Let g be a continuous, non-decreasing function on $\mathbb{R}$. For each $[a, b]$, define $\phi([a, b]) = g(b) - g(a)$. Let
   $$\mu(E) = \inf\left\{\sum_k \phi(I_k) : \ E \subset \bigcup_k I_k, I_k \text{ closed, bounded intervals}\right\} .$$

   (a) Show that $\mu([a, b]) = \phi([a, b])$.

   (b) Show that μ is a Borel measure.

4. (20 pts) Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$.

   (a) Suppose that Φ is continuous on $\mathbb{R}$. Is it true that $\Phi(E)$ is always Lebesgue measurable? Prove this or give a counter-example (and prove that it is a counter-example).

   (b) Suppose that
   $$|\Phi(y) - \Phi(x)| \leq L|x - y| \ , \quad \forall \, x, y \in E,$$
   for some positive constant L. Show that $\Phi(E)$ is Lebesgue measurable.

5. (15 pts) Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue measurable function with $f(x) > 0$ almost everywhere. Suppose (E_k) is a sequence of measurable sets in $[0, 1]$ with the property that $\int_{E_k} f \, d\mathcal{L} \to 0$ as $k \to \infty$, where $\mathcal{L}$ denotes the Lebesgue measure restricted on $[0, 1]$. Prove that $\mathcal{L}(E_k) \to 0$ as $k \to \infty$.

6. (15 pts) Let μ_1 and μ_2 be two measures on a measurable space $(X, \mathcal{M})$. Define
   $$\mu(E) = \inf\{\mu_1(E \backslash F) + \mu_2(E \cap F) : \ F \in \mathcal{M}\}$$
   for $E \in \mathcal{M}$. Prove that μ is a measure on $(X, \mathcal{M})$.