

**The Chinese University of Hong Kong**  
**MATH5011 Real Analysis I 2020-2021**  
**Final Examination**

December 11, 2020, 9:30am–12:30pm (noon)

**Instructions**

- This is an open-book examination. You may **ONLY** refer to printed/written materials during the examination. Assessing information on the internet is not allowed.
- You are allowed to use a calculator in the approved list during the examination.
- You shall take the examination in isolation and shall not communicate with any person during the examination other than the course teacher(s) concerned. Please kindly be reminded of the following regulations enforced by the university:

**Honesty in Academic Work:** *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*

- There are a total of 100 points.
- Answer **ALL** questions.
- Show all steps clearly in your working. **NO** point will be given if sufficient justification is not provided.
- Only handwritten answers on papers or electronic devices will be accepted. Typed answer will **NOT** be accepted.
- Please follow the instruction of submission below:
  1. Write your answers on papers or electronic devices. Only handwritten answers will be accepted and typed answer will **NOT** be accepted.
  2. Scan or take photos of your work (if you write on papers).
  3. Combine your work into a single pdf file.
  4. Name the pdf file by your student id (e.g. 1155123456.pdf).
  5. You must upload the pdf file to Blackboard if you are students from CUHK, or send the pdf file to me via email (djfeng@math.cuhk.edu.hk) if your are from other universities, before 12:30pm (noon), Dec 11, 2020. Mark deduction will be made for late submission.
  6. Please check your file carefully to make sure no missing pages.

Answer all 6 questions.

1. (20 marks) Let  $\{A_k\}_{k=1}^{\infty}$  be a sequence of measurable sets in a measure space  $(X, \mathcal{M}, \mu)$ . Define

$$B = \{x \in X : x \in A_k \text{ for all except finitely many } k\}.$$

Prove that

- (i)  $B$  is measurable, and
  - (ii)  $\mu(B) \leq \liminf_{k \rightarrow \infty} \mu(A_k)$ .
2. (10 marks) Let  $f$  be a real-valued function on  $\mathbb{R}$ . Show that the following set

$$C(f) := \{x \in \mathbb{R} : f \text{ is continuous at } x\}$$

is a Borel set.

3. (20 marks) Let  $X_1$  be a proper, closed subspace of a Hilbert space  $X$ . Let  $x_0 \in X$  be a point lying outside  $X_1$ . Prove that there exists a unique  $z \in X_1$  such that

$$\|z - x_0\| \leq \|y - x_0\|, \quad \forall y \in X_1.$$

4. (20 marks) Let  $f_n \in L^1(\mathbb{R})$  for  $n = 1, 2, \dots$ . Suppose that  $\int_{-\infty}^{\infty} |f_n(x)| dx < \frac{1}{n^2}$ . Prove that  $f_n(x) \rightarrow 0$  for  $\mathcal{L}$ -a.e.  $x \in \mathbb{R}$ , as  $n \rightarrow \infty$ , where  $\mathcal{L}$  stands for the Lebesgue measure on  $\mathbb{R}$ .
5. (15 marks) Suppose that  $(g_n)$  is a sequence of positive continuous functions on  $[0, 1]$ ,  $\mu$  is a positive Borel measure on  $[0, 1]$  and that (i)  $\lim_{n \rightarrow \infty} g_n(x) = 0$  for  $\mathcal{L}$ -a.e.  $x \in [0, 1]$ ; (ii)  $\int_0^1 g_n(x) dx = 1$  for all  $n$  and (iii)  $\lim_{n \rightarrow \infty} \int_0^1 f g_n dx = \int_0^1 f d\mu$  for every continuous function  $f \in C[0, 1]$ . Does it follow that  $\mu$  is mutually singular with respect to the Lebesgue measure  $\mathcal{L}$ ? Prove this or give a counter-example (and prove that it is a counter-example).
6. (15 marks) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} |\sin(nx)|, \quad x \in \mathbb{R}.$$

- (a) Show that  $f$  is continuous on  $\mathbb{R}$ .
- (b) Let  $x \in \mathbb{R}$ . Show that  $f$  is differentiable at  $x$  if and only if that  $x/\pi$  is an irrational number.