TA's remarks on 5011 homework 9

1. In this 13-question homework, you can gain 10/13 marks from each answered question. On the other hand, there is mark deduction if your Hw6-9 are overdue. The arrangement is as follows.

No. of late submittion	Mark deduction
1	0.5
2	1.5
3	3
4	5

2. We make some annotations to the solution on the coming pages.

Solution to MATH5011 homework 9

(2) Consider $L^p(\mu)$, $0 < p < 1$. Then $\dfrac{1}{q} + \dfrac{1}{p} = 1$, $q < 0$.

(a) Prove that $\|fg\|_1 \geq \|f\|_p \|g\|_q$.

Solution.

(a) If $\|g\|_q = 0$, then plainly the desired inequality holds. Similarly, if $\mu([|fg| = \infty]) > 0$, then $\|fg\|_1 = \infty$ and the inequality holds. Also, if $\mu([|g| = 0]) > 0$, then as $q < 0$, we have $\int_X |g|^q \geq \int_{[|g|=0]} |g|^q = \infty$, whence $\|g\|_q = 0$. Thus, we may assume $\|g\|_q \neq 0$, $|fg| < \infty$ a.e., and $|g| > 0$ a.e.. It follows that $|f|^p = |fg|^{1/\widetilde{p}} |g|^{-1/\widetilde{p}}$ a.e., where $\widetilde{p} := \dfrac{1}{p}$. Let $\widetilde{q} := \dfrac{1}{1-p} = \dfrac{\widetilde{p}}{\widetilde{p}-1}$ be the conjugate exponent of $\widetilde{p}$. Applying the Hölder's inequality we have

$$\begin{aligned}
\||f|^p\|_1 &= \left\| |fg|^{1/\widetilde{p}} |g|^{-1/\widetilde{p}} \right\|_1 \\
&\leq \left\| |fg|^{1/\widetilde{p}} \right\|_{\widetilde{p}} \cdot \left\| |g|^{-1/\widetilde{p}} \right\|_{\widetilde{q}} \\
&= \|fg\|_1^{1/\widetilde{p}} \left\| |g|^{-1/(\widetilde{p}-1)} \right\|_1^{(\widetilde{p}-1)/\widetilde{p}} \\
&= \|fg\|_1^p \left\| |g|^{-p/(1-p)} \right\|_1^{1-p}, \text{ so} \\
\||f|^p\|_1^{1/p} &\leq \|fg\|_1 \left\| |g|^{-p/(1-p)} \right\|_1^{1/p-1} \\
&= \|fg\|_1 \||g|^q\|_1^{-1/q}, \text{ or} \\
\|f\|_p &\leq \|fg\|_1 \|g\|_q^{-1}.
\end{aligned}$$

If $\|g\|_q = \infty$, then the above gives $\|f\|_p = 0$ and the result follows. Else, we have $0 < \|g\|_q < \infty$, so we obtain the result by multiplying both sides by $\|g\|_q$.

(3) Let X be a metric space consisting of infinitely many elements and μ a Borel measure on X such that $\mu(B) > 0$ on any metric ball (i.e. $B = \{x : d(x, x_0) < \rho\}$ for some $x_0 \in X$ and $\rho > 0$. Show that $L^\infty(\mu)$ is non-separable.

Suggestion: Find disjoint balls $B_{r_j}(x_j)$ and consider $\left\{ \sum_{n=1}^\infty a_n \chi_{B_{r_j}(x_j)} : (a_1, a_2, \ldots,) \in \{0,1\}^{\mathbb{N}} \right\}$.

Partial Solution. To find such $B_{r_j}(x_j)$, we may use the following idea suggested by a student. Let $S := \{y_1, y_2, \ldots\}$ be a countably infinite subset of X.

If S has no limit point in S, then we take $x_i := y_i$ and define $\{r_i\}$ inductively as follows. After defining $r_1, \ldots, r_{N-1}$, we pick $r_N > 0$ to be such that $B(x_N, 4r_N) \cap S = \{x_N\}$ and $r_N < r_{N-1}$. If $\xi \in B(x_N, r_N) \cap B(x_i, r_i)$ for some $i < N$, then

$$d(x_N, x_i) \leq d(x_N, \xi) + d(\xi, x_i) \leq r_N + r_i \leq 2r_i,$$

whence $x_N \in B(x_i, 4r_i)$, which is a contradiction.

Else if S has a limit point $Y \in S$, then we define $\{(x_i, r_i)\}$ inductively as follows. After defining (x_1, r_1), ..., (x_{N-1}, r_{N-1}), we pick $x_N \in S$ and $r_N > 0$ to be such that:

$$\begin{cases} 4r_N < d(x_N, Y) < d(x_i, Y) - 2r_i \text{ for all } i < N \\ r_N < r_{N-1}. \end{cases}$$

If $\xi \in B(x_N, r_N) \cap B(x_i, r_i)$ for some $i < N$, then

$$d(x_i, Y) \leq d(x_i, \xi) + d(\xi, x_N) + d(x_N, Y) \leq r_i + r_N + (d(x_i, Y) - 2r_i) < d(x_i, Y),$$

which is a contradiction.

(4) Show that $L^1(\mu)' = L^\infty(\mu)$ provided $(X, \mathfrak{M}, \mu)$ is σ-finite, i.e., $\exists X_j$, $\mu(X_j) < \infty$, such that $X = \bigcup X_j$.
Hint: First assume $\mu(X) < \infty$. Show that $\exists g \in L^q(\mu)$, $\forall q > 1$, such that

$$\Lambda f = \int fg\, d\mu, \quad \forall f \in L^p,\ p > 1.$$

Next show that $g \in L^\infty(\mu)$ by proving the set $\{x : |g(x)| \geq M + \varepsilon\}$ has measure zero $\forall \varepsilon > 0$. Here $M = \|\Lambda\|$.

Solution.

Please refer to Rudin's *Real and Complex Analysis* Theorem 6.16. Alternately, we have the following two-step proof.
(...omitted...)

Step 2. $\mu(X) = \infty$.
The previous conclusion can be extended to the case that $\mu(X) = \infty$ but X is σ-finite. Then

$$X = \bigcup_{j=1}^{\infty} X_j$$

with $\mu(X_j)$ finite and with $X_j \cap X_k$ empty whenever $j \neq k$. Any $L^1(X)$ function f can be written as

$$f(x) = \sum_{j=1}^{\infty} f_j(x)$$

where $f_j = \chi_j f$ and χ_j is the characteristic function of X_j. $f_j \mapsto \Lambda f_j$ is then an element of $L^1(X_j)'$, and hence there is a function $v_j \in L^\infty(X_j)$ such that $\Lambda f_j = \int_{X_j} v_j f_j\, d\mu = \int_{X_j} v_j f\, d\mu$. The important point is that each v_j is bounded in $L^\infty(X_j)$ by the *same* $\|\Lambda\|$. Moreover, the function v, defined on all of X by $v(x) = v_j(x)$ for $x \in X_j$, is clearly measurable and bounded by $\|\Lambda\|$. Given any $f \in L^1(X)$, we have $\infty > \int_X |f|\, d\mu = \sum_j \int_X |f_j|\, d\mu = \sum_j \|f_j\|_1$. This implies $\|f - \sum_1^n f_j\|_1 \leq \sum_{j>n} \|f_j\|_1 \to 0$ as $n \to \infty$. Since Λ is continuous on $L^1(X)$, it follows that $\Lambda(f) = \lim_n \Lambda(\sum_1^n f_j) = \lim_n \sum_1^n \int_{X_j} v_j f\, d\mu = \int_X v f\, d\mu$.
If there exist $v, w \in L^\infty(X)$ such that

$$\Lambda f = \int_X vf\, d\mu = \int_X wf\, d\mu, \quad \forall f \in L^1(X),$$

then

$$\int_X (v - w) f\, d\mu = 0, \quad \forall f \in L^1(X).$$

Suppose, on the contrary, that $(v - w) > 0$ on some $A \in \mathfrak{M}$ with $0 < \mu(A)$. Since X is σ-finite, we may assume $\mu(A) < \infty$ too. By taking $f = \chi_A$ one arrives at a contradiction. Thus, given $\Lambda \in L^1(X)$ there corresponds a unique $v \in L^\infty(X)$.

(7) Optional. Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0, 1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$, and therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fg\, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

Solution. Method 1. For any $x \in I$ take $\Lambda_x f = g(x_+) - g(x_-)$ for all f such that $f = g$ a.e. for some function g such that the two one-sided limits $g(x_+)$ and $g(x_-)$ both exist. Then $\|\Lambda_x - \Lambda_y\| \geq 1$ for $x \neq y$. With reference to the question, we can just take $x = 1/2$.

(I am not sure if I understand this method correctly. Consider $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \sin\left(\dfrac{1}{x - 0.5}\right) & \text{if } x \neq 0.5 \\ 1 & \text{otherwise.} \end{cases}$$

We have $f \in L^\infty$. Claim that given any function g such that $g = f$ a.e., $g(x_-)$ does not exist, whence $\Lambda_{0.5}(f)$ is undefined. To justify this, note that for each $n \in \mathbb{N}$, there exists an interval $(a, b) \subseteq (0.5 - 1/n,\ 0.5)$ such that $f > 0.8$ on (a, b). If $g(x) \neq f(x)$ for all $x \in (a, b)$, then $\mathcal{L}([g \neq f]) \geq \mathcal{L}(a, b) > 0$, which is a contradiction. Hence $g(x) > 0.8$ for some $x \in (0.5 - 1/n,\ 0.5)$. This shows $\overline{\lim}_{x \uparrow 0.5} g(x) \geq 0.8$. Similarly we can show $\underline{\lim}_{x \uparrow 0.5} g(x) \leq -0.8$. Hence $g(x_-)$ does not exist.

Maybe this solution means an extension of $\Lambda_{0.5}$ to L^∞ by Hahn-Banach Theorem (Rudin's Theorem 5.16). I am not so sure about this functional analysis stuff.)

Method 2. Consider $\chi_{[0,\frac{1}{2}]} \in L^\infty \setminus C(I)$, as $C(I)$ is closed subspace in L^∞, by consequence of Hahn-Banach Theorem (Rudin's Theorem 5.19), there is non-zero bounded linear functional Λ on L^∞ which is zero on $C(I)$. Let $f_0 \in L^\infty$ be such that $\Lambda(f_0) \neq 0$.

Suppose there is $g \in L^1(m)$ that satisfies $\Lambda f = \displaystyle\int_I fg\, dm$ for every $f \in L^\infty$. Let $\varepsilon > 0$. By Hw2 Q10, there exists $\delta > 0$ such that $\int_A |g|\, dm < \varepsilon$ whenever $m(A) < \delta$. By lecture notes Theorem 2.12 (Lusin's Theorem), there exists $h \in C(I)$ such that

$$\begin{cases} m([f_0 \neq h]) < \delta \\ \|h\|_\infty \leq \|f_0\|_\infty . \end{cases}$$

Since $\Lambda(h) = 0$, we have

$$\left| \int_I f_0 g dm \right| = \left| \int_I (f_0 - h) g dm \right| \leq 2 \|f_0\|_\infty \int_{[f_0 \neq h]} |g|\, dm \leq 2 \|f_0\|_\infty \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it follows that $\Lambda(f_0) = 0$ which is impossible.

(8) Prove Brezis-Lieb lemma for $0 < p \leq 1$.
Hint: Use $|a + b|^p \leq |a|^p + |b|^p$ in this range.

Note that by the hint $||f_n|^p - |f_n - f|^p - |f|^p| = |f|^p + (|f_n - f|^p - |f_n|^p)$. The expression in the parenthesis is $\leq |-f|^p = |f|^p$ by using the hint once more. As in the proof of the Brezis-Lieb Lemma in the lecture notes, $|f|^p$ is integrable by Fatou's lemma.

(11) We have the following version of Vitali's convergence theorem. Let $\{f_n\} \subset L^p(\mu)$, $1 \leq p < \infty$. Then $f_n \to f$ in L^p-norm if and only if

(i) $\{f_n\}$ converges to f in measure,

(ii) $\{|f_n|^p\}$ is uniformly integrable, and

(iii) $\forall \varepsilon > 0$, there exists a measurable E, $\mu(E) < \infty$, such that $\displaystyle\int_{X \setminus E} |f_n|^p\, d\mu < \varepsilon$, $\forall n$.

I found this statement from PlanetMath. Prove or disprove it.

Solution. ($\Leftarrow$) Let $\varepsilon > 0$. By (iii), there exists a set E of finite measure such that

$$\int_{\widetilde{E}} |f_n|^p < \varepsilon.$$

Since $\{f_n\}$ converges to f in measure, there is a subsequence $\{f_{n_k}\}$ which converges to f pointwisely a.e.. By Fatou's Lemma,

$$\int_{\widetilde{E}} |f|^p \leq \varepsilon.$$

By (ii), there exists $\delta > 0$ such that whenever $\mu(A) < \delta$,

$$\int_A |f_n|^p < \varepsilon;$$

Then whenever $\mu(A) < \delta$, we have

$$\int_A |f|^p \leq \varepsilon$$

because there is a subsequence $\{f_{n_k}\}$ which converges to f pointwisely a.e. and we can apply Fatou's Lemma. Suppose E is of positive measure first. By (i), there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mu\{x \in E : \left|(f_n - f)(x)\right|^p \geq \frac{\varepsilon}{\mu(E)}|\} < \delta.$$

Now, for $n \geq \mathbb{N}$, define $A_n = \{x \in E : |(f_n - f)(x)|^p \geq \frac{\varepsilon}{\mu(E)}\}$ and $B_n = E \setminus A_n$. Using the Vinogradov notation introduced in remark 5 and noting that $|f_n - f|^p \leq (|f_n| + |f|)^p \leq (2|f_n|)^p + (2|f|)^p \ll |f_n|^p + |f|^p$, we have

$$\begin{aligned}
\int |f_n - f|^p &= \int_{\widetilde{E}} |f_n - f|^p + \int_{A_n} |f_n - f|^p + \int_{B_n} |f_n - f|^p \\
&\ll \int_{\widetilde{E}} |f_n|^p + \int_{\widetilde{E}} |f|^p + \int_{A_n} |f_n|^p + \int_{A_n} |f|^p + \int_{B_n} |f_n - f|^p \\
&\ll \varepsilon + \int_{B_n} |f_n - f|^p \\
&\ll \varepsilon.
\end{aligned}$$

On the other hand, if E is of zero measure, then we also have

$$\int |f_n - f|^p = \int_{\widetilde{E}} |f_n - f|^p \ll \int_{\widetilde{E}} |f_n|^p + \int_{\widetilde{E}} |f|^p \ll \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $f_n \to f$ in L^p-norm.

($\Rightarrow$) Please refer to `https://planetmath.org/ProofOfVitaliConvergenceTheorem` for detail.

(14) Let $\{f_n\}$ be bounded in $L^p(\mu)$, $1 < p < \infty$. Prove that if $f_n \to f$ a.e., then $f_n \rightharpoonup f$. Is this result still true when $p = 1$?

Solution.

(...omitted...)

An alternate approach is, using the L^p-boundedness and lecture notes Theorem 4.27, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ weakly converges to some $g \in L^p(\mu)$. Let $E_K := \{f_{n_k} : k \geq K\}$. By lecture notes Proposition 4.26, for each $K \geq 1$, there exists a convex combination F_K of functions from E_K such that $\|F_K - g\|_p \leq 1/K$. By lecture notes Corollary 4.13, we have a subsequence $\{F_{K_\ell}\}$ of $\{F_K\}$ converges pointwise to g a.e.. On the other hand, F_K converges pointwise to f a.e., because if we write

$$F_K = \theta_1 f_{a_1} + \cdots + \theta_m f_{a_m},$$

where $\theta_1 + \cdots + \theta_m = 1$ and $a_1 < \cdots < a_m$, then by the definition of E_K, we have $\lim_{K\to\infty} a_1 = \infty$, whence by $f_n \to f$ a.e., we have for a.e. x,

$$|F_K(x) - f(x)| = \left|\sum_1^m \theta_i(f_{a_i}(x) - f(x))\right| \leq \sum_1^m \theta_i |f_{a_i}(x) - f(x)| \leq \max_i |f_{a_i}(x) - f(x)| \to 0$$

as $K \to \infty$. So $g = f$ a.e.. We have shown that every weakly convergent subsequence of $\{f_n\}$ must converge weakly to f. Now, suppose that f_n does not converge weakly to f. There are $\rho > 0$ and $g \in L^q$, such that

$$\left|\int f_{n_k} g d\mu - \int f g d\mu\right| > \rho\ , \quad \forall n_k$$

for some subsequence f_{n_k}. But we can find a subsequence from this subsequence which converges weakly to f, contradiction holds.

For p=1, the result is false by the last problem.