TA's remarks on 5011 homework 6

1. The mark distribution for Hw6 is:
   Q2, 4, 5, 6 (2.5 marks each).

2. In the solution to Q4, note that $\{0,1\} \subseteq K$, whence $G = [0,1] \setminus K = (0,1) \cap K^c$ is an open set in $\mathbb{R}$.

3. An alternative approach to Q4 is as follows. Since $\mathbb{Q} \cap [0,1]$ is a set of $\mathcal{L}^1$-measure zero, by outer regularity there exists an open set $\mathcal{O} \supseteq \mathbb{Q} \cap [0,1]$ such that $\mathcal{L}^1(\mathcal{O}) \leq \varepsilon/2$. Let $U := \mathcal{O} \cap (0,1)$, so that $U \subseteq [0,1]$ is an open set dense in $[0,1]$, and $\mathcal{L}^1(U) \leq \mathcal{L}^1(\mathcal{O}) \leq \varepsilon/2$.

   Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \mathcal{L}^1(U \cup (0,x))$. If $h \geq 0$ and $x, x+h \in [0,1]$, then $|f(x+h) - f(x)| \leq \mathcal{L}^1([x, x+h]) = h$. This shows that f is continuous. Since $f(0) = \mathcal{L}^1(U) \leq \varepsilon/2$ and $f(1) = \mathcal{L}^1([0,1]) = 1$, by the intermediate value theorem, there exists $x_0 \in [0,1]$ such that $f(x_0) = \varepsilon$. As a result, the set $G := U \cup (0, x_0)$ satisfies the desired properties.

   We remark that the set $\mathcal{O}$ at the beginning can also be taken as

   $$\bigcup_{r_n \in \mathbb{Q} \cap [0,1]} (r_n - \frac{1}{16}\frac{\varepsilon}{2^n},\ r_n + \frac{1}{16}\frac{\varepsilon}{2^n}).$$

4. Q5 gives a glimpse of how fractal geometry plays a role in the study of number theory (numeration systems and beta-expansions to be specific). Here we present an alternative solution. The interested set A can be constructed like the Cantor set. In each stage, we divide the remaining intervals into 10 parts of equal length, and we remove the 5th. In this way, at the k-th stage, each interval is of length $1/10^k$. Each interval from the $(k-1)$-th stage produces $10 - 1 = 9$ intervals for the k-th stage, so by induction, there are in total 9^k intervals at the k-th stage. If A_k denotes the set for the k-th stage, then

   $$\mathcal{L}^1(A) = \lim_k \mathcal{L}^1(A_k) = \lim_k \left(\frac{9}{10}\right)^k = 0.$$

   This fits the probabilistic heuristic that the probability for a random $x \in [0,1]$ to have no digit 4 in the first k decimal places is $(9/10)^k$.

5. We may think of Q8(a) this way. It appears that for any open and bounded intervals I, J, the function $x \mapsto \mathcal{L}^1((x+I) \cap J)$ is continuous. Since an open set in $\mathbb{R}$ can be written as a countable disjoint union of open intervals, therefore by approximation we should be able to get the result.

   To be more precise, write V as a disjoint union of open intervals, $V =: \bigcup_{j=1}^\infty J_j$ (some J_j may be empty). As $\mathcal{L}^1(V) < \infty$, each J_j is bounded. Given an open and bounded interval I (which may also be empty), define $f_{I,n}, f_I : \mathbb{R} \to \mathbb{R}$ by

   $$f_{I,n}(x) := \sum_{j=1}^n \mathcal{L}^1((x+I) \cap J_j), \quad f_I(x) := \mathcal{L}^1((x+I) \cap V).$$

   Since

   $$|f_{I,n}(x) - f_I(x)| = \left| \sum_{j=1}^n \mathcal{L}^1((x+I) \cap J_j) - \sum_{j=1}^\infty \mathcal{L}^1((x+I) \cap J_j) \right| \leq \sum_{j=n+1}^\infty \mathcal{L}^1(J_j),$$

   and $\sum_1^\infty \mathcal{L}^1(J_j) = \mathcal{L}^1(V) < \infty$, we see that $f_{I,n}$ converges uniformly to f_I. As each $f_{I,n}$ is continuous, so is f_I.

Next, write U as a disjoint union of open intervals, $U =: \bigcup_{i=1}^{\infty} I_i$. Again, as $\mathcal{L}^1(U) < \infty$, each I_i is bounded. Define $g_n, g : \mathbb{R} \to \mathbb{R}$ by

$$g_n(x) := \sum_{i=1}^{n} f_{I_i}(x), \quad g(x) := \mathcal{L}^1((x+U) \cap V).$$

As

$$|g_n(x) - g(x)| = \left| \sum_{i=1}^{n} \mathcal{L}^1((x+I_i) \cap V) - \sum_{i=1}^{\infty} \mathcal{L}^1((x+I_i) \cap V) \right| \leq \sum_{i=n+1}^{\infty} \mathcal{L}^1(x+I_i) = \sum_{i=n+1}^{\infty} \mathcal{L}^1(I_i),$$

we see that g_n converges uniformly to g. Since each g_n is continuous, so is g, which was to be demonstrated.