TA's remarks on 5011 homework 5

1. The mark distribution for Hw5 is:
   Q1, 2 (5 marks each).

2. In Q1, to show that the set $T(E)$ is measurable, we may use the result of Q2. On the other hand, to show that $T(\mathbb{R}^n)$ is of $\mathcal{L}^n$-measure zero if $T : \mathbb{R}^n \to \mathbb{R}^n$ is singular, we may argue as follows. Let $\{\mathbf{y_1}, \cdots, \mathbf{y_k}\}$ be an orthonormal basis for the vector space $T(\mathbb{R}^n)$, and $\{\mathbf{y_1}, \cdots, \mathbf{y_k}, \mathbf{y_{k+1}}, \cdots, \mathbf{y_n}\}$ an orthonormal basis for $\mathbb{R}^n$. Define an invertible linear transformation $U : \mathbb{R}^n \to \mathbb{R}^n$ by

$$U(\sum_1^n a_i\mathbf{e_i}) := \sum_1^n a_i\mathbf{y_i},$$

   where $\{\mathbf{e_i}\}_{i=1}^n$ denotes the standard basis for $\mathbb{R}^n$. It follows from the "invertible case" in Q1 that there exists a constant $C \geq 0$ such that $\mathcal{L}^n(U(E)) = C\mathcal{L}^n(E)$ for all Lebesgue measurable set E. As a result,

$$\mathcal{L}^n(T(\mathbb{R}^n)) = \mathcal{L}^n(U(\mathbb{R}^k \times \{0\}^{n-k})) = C\mathcal{L}^n(\mathbb{R}^k \times \{0\}^{n-k}) = C \lim_{\ell\to\infty} \mathcal{L}^n([-\ell, \ell]^k \times \{0\}^{n-k}) = 0.$$

3. In the solution to Q2, the approach used also appears in Rudin's *Real and Complex Analysis* Lemma 7.25 and Theorem 7.26. On the other hand, to show that $\Phi(U)$ is of $\mathcal{L}^n$-measure zero if U is, by the results in lecture notes Ch3 we may argue as follows. Given $f, g \geq 0$, write "$f \ll g$" (the Vinogradov notation) if there exists a constant $C \geq 0$ such that $f \leq Cg$. Treating $\mathcal{L}^n$ and $\mathcal{H}^n$ (the n-dimensional Hausdorff measure) as outer measures, we have

$$\begin{aligned} \mathcal{L}^n(\Phi(U)) &\ll \mathcal{H}^n(\Phi(U)) \quad \text{(Proposition 3.10)} \\ &\ll \mathcal{H}^n(U) \quad \text{(Proposition 3.11)} \\ &\ll \mathcal{L}^n(U) \quad \text{(Proposition 3.10)} \\ &= 0. \end{aligned}$$

   One may inspect Proposition 3.11 or Rudin's Lemma 7.25 for the essential arguement.