TA's remarks on 5011 homework 4

1. The mark distribution for Hw4 is:
   Q2 (3 marks); Q4 (3 marks); Q6 (4 marks).

2. We use Caratheodory's criterion multiple times in this exercise. A reference can be found in Royden's *Real analysis* in the name of "Carathéodory outer measure" and in the following more general setting:

   > Let X be a set of points and Γ a set of real-valued functions on X. It is often of interest to know conditions under which an outer measure μ^* will have the property that every function in Γ will be measurable ...

3. In the solution to Q1(c), the idea is to consider set complement and make use of the result of Q1(b). The solution also makes use of the result of Q1(a) implicitly in many places for countable additivity. It may be rewritten as

   > We decompose $\mathbb{R}^n = \bigcup_k E_k$ where $E_1 := [-1,1]^n$ and $E_k := [-k,k]^n \setminus (-(k-1), k-1)^n$ for $k \geq 2$. Fix a measurable A and an $\varepsilon > 0$. Define $A_k := A \cap E_k$. By Q1(b) and the argument in Lecture notes proof of Proposition 2.10(a), we can find an open $G_k \supseteq A_k^c$ such that $\mathcal{L}^n(G_k \setminus A_k^c) < \varepsilon/2^k$. Define $K_k := E_k \cap G_k^c$. As K_k is closed and bounded, it is a compact set. We have $\mathcal{L}^n(A_k \setminus K_k) = \mathcal{L}^n(A_k \cap (E_k^c \cup G_k)) = \mathcal{L}^n(A_k \cap G_k) = \mathcal{L}^n(G_k \setminus A_k^c) < \varepsilon/2^k$. We also have $K_k \subseteq G_k^c \subseteq A_k$.
   >
   > By Hw3 solution to Ex8(b), we know that the boundary of a cube is of $\mathcal{L}^n$-measure zero. Therefore, we have $\mathcal{L}^n(\cup A_k) = \sum \mathcal{L}^n(A_k)$ and $\mathcal{L}^n(\cup K_k) = \sum \mathcal{L}^n(K_k)$, even though they may not be disjoint unions.
   >
   > If $\mathcal{L}^n(A) < \infty$, we can fix some large N such that $\sum_N^\infty \mathcal{L}^n(A_k) < \varepsilon$. The compact set $K := \cup_{k=1}^N K_k$ satisfies $K \subseteq A$ and $\mathcal{L}^n(A \setminus K) \leq \mathcal{L}^n(\cup_{k=1}^N (A_k \setminus K_k)) + \varepsilon \leq 2\varepsilon$. Else if $\mathcal{L}^n(A) = \infty$, then for each $M > 0$, we can find some large N such that $\sum_1^N \mathcal{L}^n(A_k) > M + \varepsilon$. Then $K := \cup_{k=1}^N K_k$ satisfies $K \subseteq A$ and $M + \varepsilon < \sum_1^N \mathcal{L}^n(A_k) = \sum_1^N (\mathcal{L}^n(K_k) + \mathcal{L}^n(A_k \setminus K_k)) \leq \sum_1^N \mathcal{L}^n(K_k) + \varepsilon = \mathcal{L}^n(K) + \varepsilon$. The result follows.

4. In the solution to Q2, we should first show that the boundary of a cube has zero μ-measure, so that we can use countable additivity for any almost disjoint union of cubes without being bothered by their non-empty intersections. An approach can be found in Hw3 solution to Ex8(b). Alternatively, we can consider covering given by the collection ~~$\{[k/2^\ell, (k+1)/2^\ell)^n : \ell \in \mathbb{N}, k \in \mathbb{Z}\}$~~ $\{\Pi_{i=1}^n [k_i/2^\ell, (k_i+1)/2^\ell) : \ell \in \mathbb{N}, (k_1, \cdots, k_n) \in \mathbb{Z}^n\}$ (the intervals are half closed, half open). Every nonempty open set in $\mathbb{R}^n$ is a countable disjoint union of sets from this collection, c.f. Rudin's *Real and Complex Analysis* section 2.19 Euclidean Spaces.

5. In the later part of the solution to Q2, we can drop the assumption that E is bounded, because lecture notes Ch2 Proposition 2.10 still works for unbounded E.

6. In Q2, as an alternative approach, after showing that $\mu = C \cdot \mathcal{L}^n$ on open sets, to show that $\mu = C \cdot \mathcal{L}^n$ on Borel sets, we can show that μ is outer regular, for then as $\mathcal{L}^n$ is outer regular, we have

   $$\mu(E) = \inf_{G \supseteq E,\ G \text{ open}} \{\mu(G)\} = \inf_{G \supseteq E,\ G \text{ open}} \{C \cdot \mathcal{L}^n(G)\} = C \cdot \mathcal{L}^n(E).$$

   To show μ is outer regular, we may use lecture notes Ch2 Proposition 2.11 and then Proposition 2.9.

7. We will revisit the solution to Q3 and Q4 in lecture note Ch3 when studying Hausdorff measure. In the solution to Q3, note that $\inf \emptyset = \infty$, whence "$\mu_\delta(A) = \infty$" includes the possibility that there is no "δ-covering" for A. Also, note that as $\delta \downarrow 0$, we have $\mu_\delta \uparrow \mu$ instead of $\mu_\delta \downarrow \mu$.

8. In Q4, we have the following example[1]. In $\mathbb{R}$, consider $s := 0$ such that

$$\rho(E) := \begin{cases} 0 & \text{if } E = \emptyset \\ 1 & \text{if } E \neq \emptyset \text{ and } \operatorname{diam} E = 0 \\ 1 & \text{if } \operatorname{diam} E > 0. \end{cases}$$

Let $\delta := 2$, $A := \{0\}$ and $B := \{1\}$. Then

$$\mu_2(A \cup B) = 1 \neq 1 + 1 = \mu_2(A) + \mu_2(B).$$

As another example[2], in $\mathbb{R}^2$ with $s := 1$, $\delta := 10$, $A := [0,1] \times \{0\}$, $B := [0,1] \times \{0.1\}$, we have

$$\begin{aligned} \mu_{10}(A) &= \inf\left\{\sum_k \operatorname{diam}(C_k) : A \subseteq \bigcup_k C_k, \operatorname{diam}(C_k) \leq 10\right\} \\ &= \inf\left\{\sum_k \operatorname{diam}(C_k) : A \subseteq \bigcup_k C_k, \operatorname{diam}(C_k) \leq 10, (x,y) \in C_k \Rightarrow y = 0\right\} \\ &= \mathcal{L}([0,1]) = 1. \end{aligned}$$

Similarly $\mu_{10}(B) = 1$. On the other hand, as $\operatorname{diam}([0,1] \times \{0, 0.1\}) = \sqrt{1.01} \leq 10$, we have

$$\mu_{10}(A \cup B) \leq \sqrt{1.01} < 2 = \mu_{10}(A) + \mu_{10}(B).$$

9. In Q6, instead of going through the construction steps of Riesz measure, we can make a guess first and then justify our answer by some uniqueness result. The argument is like the following. Define a Borel measure μ on $\mathbb{R}$ and show that it satisfies

$$\Lambda(f) = \int_{\mathbb{R}} f \, d\mu$$

for all $f \in C_c(\mathbb{R})$ and blah blah blah. Conclude by using a uniqueness argument/theorem that μ is the Riesz measure for Λ.

Such uniqueness argument can be found in e.g. Rudin's *Real and Complex Analysis* Theorem 2.14 and the discussion therein.

[1] A student suggests this example.

[2] c.f. `https://math.stackexchange.com/questions/2942022/`. A student suggests this example.