TA's remarks on 5011 homework 2

1. The mark distribution for Hw2 is:
   Q1, 4, 7, 8, 10 (2 marks each).

2. Consider a measure space $(X, \mathcal{M}, \mu)$ where $X \neq \emptyset$ and $\mu(X) = 0$, and the integrable function $f : X \to \overline{\mathbb{R}}$ defined by $f(x) = \infty$ for all $x \in X$. This example helps us examine our answers to Q4 and Q10.

3. The result of Q10 has been used in the solution to Q5 in the following way. Suppose we have shown that
   $$\int_G f^+ = \int_G f^-$$
   for all open G, and we want to show that
   $$\int_E f^+ = \int_E f^-$$
   for a Lebesgue measurable set E. For a fixed $\varepsilon > 0$, there exists open set $G_\varepsilon \supseteq E$ such that
   $$\mathcal{L}(G_\varepsilon \setminus E) < \delta,$$
   where δ is obtained from Q10 so that
   $$\int_A |f| < \varepsilon$$
   whenever A is a Lebesgue measurable set with $\mathcal{L}(A) < \delta$. Therefore,
   $$\left| \int_E f^+ - \int_E f^- \right| = \left| \int_{G_\varepsilon} f^+ - \int_{G_\varepsilon \setminus E} f^+ - \int_{G_\varepsilon} f^- + \int_{G_\varepsilon \setminus E} f^- \right| \leq \int_{G_\varepsilon \setminus E} |f| < \varepsilon.$$
   Since $\varepsilon > 0$ is arbitrary, the result follows.

4. If $f : X \to \mathbb{R}$ is a measurable function in $(X, \mathcal{M}, \mu)$, and $g : X \to \mathbb{R}$ is a function such that $g = f$ a.e., then it is not necessarily true that g is measurable. For example, suppose $U \in \mathcal{M}$, $\mu(U) = 0$, $N \subseteq U$, and $N \notin \mathcal{M}$. Define $f, g : X \to \mathbb{R}$ by $f(x) := 1$ for all $x \in X$, and
   $$g(x) := \begin{cases} 1 & \text{if } x \in X \setminus U \\ 2 & \text{if } x \in U \setminus N \\ 3 & \text{if } x \in N. \end{cases}$$
   Then $g = f$ a.e., but g is not measurable because $g^{-1}\{3\} \notin \mathcal{M}$. What bothers us is that μ is not complete. On the other hand, the solution to Q9(a) works because the Lebesgue measure is complete.

5. An integrable function takes finite values a.e., but it need not be bounded a.e.. For example, consider $f : [0, 1] \to [0, \infty]$ defined by $f(x) := 1/x^{0.5}$. Note that
   $$\int_0^1 f dx = \left[2x^{0.5}\right]_0^1 < \infty.$$
   With this example in mind, we consider the following alternative solution to Q10:

   Fix $\varepsilon_0 > 0$. Given f is integrable, we have
   $$\infty > \int_X |f| \, d\mu = \int_{\{|f| < \infty\}} |f| \, d\mu = \sum_{n=0}^{\infty} \int_{\{n \leq |f| < n+1\}} |f| \, d\mu$$

by the monotone convergence theorem, whence there exists $N > 0$ such that

$$\begin{aligned}
\varepsilon_0 &> \sum_{n=N}^{\infty} \int_{\{n \le |f| < n+1\}} |f| \, d\mu \\
&= \int_{\{N \le |f| < \infty\}} |f| \, d\mu \quad \text{(monotone convergence theorem again)} \\
&= \int_{\{N \le |f|\}} |f| \, d\mu.
\end{aligned}$$

Define $\delta := \varepsilon_0 / N$. For any measurable set E with $\mu(E) < \delta$, we have

$$\int_E |f| \, d\mu = \int_{\{|f| < N\} \cap E} |f| \, d\mu + \int_{\{N \le |f|\} \cap E} |f| \, d\mu \le N\mu(E) + \varepsilon_0 \le 2\varepsilon_0,$$

which was to be demonstrated.