TA's remarks on 5011 homework 10

1. The mark distribution for Hw10 is:
   Q2 (3 marks); Q3 (3 marks); Q6(a)/Q8 (4 marks).

2. An alternative solution to Q1 is as follows. If $d\mathcal{L}^1 = hd\mu$, then for each $x \in (0,1)$, we have $h(x) = \int_{\{x\}} hd\mu = \mathcal{L}^1(\{x\}) = 0$, whence $h \equiv 0$ and $\mathcal{L}^1 \equiv 0$, which is a contradiction.

3. Different from the previous chapters, lecture notes Ch5 is about a new kind of math object called signed measure. Previous results of positive measures are not automatically extended to signed measures. For example, given a signed measure λ, we have not yet proved any result about whether $\lim_n \lambda(E_n) = \lambda(\bigcap_1^\infty E_n)$ if $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$.

4. Consequently, we may consider another solution to Q2. One approach is given by Rudin's *Real and Complex Analysis* Theorem 6.11, which makes use the result that "$\lambda \ll \mu \Rightarrow |\lambda| \ll \mu$" (lecture notes proposition 5.4). Since $|\lambda|$ is a positive measure, previous results can be applied to it.

5. Up to lecture notes section 5.2, or Rudin's section 6.17, we do not know what integration with respect to a signed measure is. When we talked about integration, it involved positive measures only. It is in lecture notes section 5.3, or Rudin's section 6.18, that we start to define

   $$\int f d\lambda := \int \left(f \cdot \frac{d\lambda}{d\,|\lambda|} \right) d\,|\lambda|$$

   if the R.H.S. makes sense. Similarly, we have not yet introduced what $L^1(\lambda)$ means when λ is a signed measure.

6. Q3 asks

   > Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Show that $\int f\, d\lambda = \int fh\, d\mu, \quad \forall f \in L^1(\lambda),\ fh \in L^1(\mu)$, where $h = \frac{d\lambda}{d\mu} \in L^1(\mu)$.

   In view of the above discussion, we may rephrase it as

   > Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Define $h := \frac{d\lambda}{d\mu} \in L^1(\mu)$. Suppose $f : X \to \overline{\mathbb{R}}$ satisfies $f \in L^1(|\lambda|)$ and $fh \in L^1(\mu)$. Show that $\int \left(f \cdot \frac{d\lambda}{d|\lambda|} \right) d\,|\lambda| = \int fh\, d\mu$.

   Note that as $\left| \frac{d\lambda}{d|\lambda|} \right| \equiv 1$ (lecture notes proposition 5.7), we have $f \cdot \frac{d\lambda}{d|\lambda|} \in L^1(|\lambda|)$.

7. Another thought is that we may try to define

   $$\int f d\lambda := \int f d\lambda^+ - \int f d\lambda^-$$

   if the two terms on the R.H.S. make sense. Here $\lambda^\pm$ represent the Jordan decomposition of λ. As a result, another way to rephrase Q3 may be

   > Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Define $h := \frac{d\lambda}{d\mu} \in L^1(\mu)$. Suppose $f : X \to \overline{\mathbb{R}}$ satisfies $f \in L^1(\lambda^+) \cap L^1(\lambda^-)$ and $fh \in L^1(\mu)$. Show that $\int f d\lambda^+ - \int f d\lambda^- = \int fh\, d\mu$.

8. In light of the previous discussion, below we present a solution to Q3 which is an unabridged work of a student.

> This requires a rigorous definition of integrals wrt signed measures, note one found in stackexchange is by using Hahn decomposition to decompose λ into positive part m_1 and negative part m_2, then one evaluates integrals separately (defining $\int f\,dm = \int f\,dm_1 - \int f\,dm_2$ retains duality between measure and functions). With above decomposition of λ to m_1, m_2. One first compute with characteristic function (of λ-measurable set, hence m_i-measurable.), then by linearity of integral (proven first to each m_i, then the difference of m_i 's.), it holds for simple functions. For characteristic function, let $E \in \mathfrak{M}$, then
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> $$\int \chi_E\,d\lambda = \int \chi_E\,dm_1 - \int \chi_E\,dm_2 = m_1(E) - m_2(E) = \lambda(E) = \int_E h\,d\mu = \int \chi_E h\,d\mu$$
>
> by Radon-Nikodym thm..
> if f is nonnegative λ-integrable, hence by definition, f is both m_1, m_2 measurable. And there exists a monotone sequence of measurable simple functions s_i convergent to f.
>
> $$\begin{aligned}\int f\,d\lambda &= \int f\,dm_1 - \int f\,dm_2\\ &= \left(\lim_{i\to\infty}\int s_i\,dm_1\right) - \left(\lim_{i\to\infty}\int s_i\,dm_2\right) = \left(\lim_{i\to\infty}\int s_i\,dm_1 - \int s_i\,dm_2\right)\\ &= \lim_{i\to\infty}\int s_i d\lambda = \lim_{i\to\infty}\int s_i h\,d\mu\\ &= \int fhd\mu\end{aligned}$$
>
> The last limit is because one already have $fh \in L^1(\mu)$ (by assumption) and hence by monotonicity $s_i h \in L^1(\mu)$ and uniformly bounded above (in magnitude) by fh, hence it follows by Dominated convergence thm. Finally for general $f \in L^1(\lambda)$ such that $fh \in L^1(\mu)$. Then
>
> $$\begin{aligned}\int f\,d\lambda &= \int f^+\,d\lambda - \int f^-\,d\lambda\\ &= \int f^+ h\,d\mu - \int f^- h\,d\mu = \int fhd\mu\end{aligned}$$

The second last line is because, by our definition,

$$\text{L.H.S.} = \int fd\lambda^+ - \int fd\lambda^- = \left(\int f^+ d\lambda^+ - \int f^- d\lambda^+\right) - \left(\int f^+ d\lambda^- - \int f^- d\lambda^-\right) = \text{R.H.S..}$$

9. We may think of the solution to Q6 two ways. Firstly, it is a solution to lecture notes proposition 5.4 rather than 5.8. Secondly, note that if $\lambda \perp \mu$ for positive and nonzero measures λ, μ on $(X, \mathfrak{M})$, then μ is concentrated on X but $\lambda(X) \neq 0$. We may refer to Rudin's proposition 6.8 for a better proof of lecture notes proposition 5.4.

10. We provide another solution to Q6 assuming the result of Q7.

    (a) We first show that "$\mu_1, \mu_2 \in M_r(X) \Rightarrow \mu_1 + \mu_2 \in M_r(X)$".
    Observe that given $E \in \mathcal{B}$ and a countable partition $\{E_j\}$ of E, we have

    $$\sum_j |(\mu_1 + \mu_2)(E_j)| \le \sum_j |\mu_1(E_j)| + \sum_j |\mu_2(E_j)| \le |\mu_1|(E) + |\mu_2|(E),$$

    whence $|\mu_1 + \mu_2|(E) \le |\mu_1|(E) + |\mu_2|(E)$ by taking sup over $\{E_j\}$ on the L.H.S..

Let $\varepsilon > 0$ and $E \in \mathcal{B}$. Since $\mu_1, \mu_2 \in M_r(X)$ and $|\mu_i|$ are finite measures, there exist open sets $G_i \supseteq E$ and compact sets $K_i \subseteq E$ such that $|\mu_i|(G_i \setminus E) \leq \varepsilon$ and $|\mu_i|(E \setminus K_i) \leq \varepsilon$. Therefore, for $G := G_1 \cap G_2$ and $K := K_1 \cup K_2$, we have

$$\begin{aligned} |\mu_1 + \mu_2|(G \setminus E) &\leq |\mu_1|(G \setminus E) + |\mu_2|(G \setminus E) \\ &\leq |\mu_1|(G_1 \setminus E) + |\mu_2|(G_2 \setminus E) \leq 2\varepsilon, \end{aligned}$$

and similarly $|\mu_1 + \mu_2|(E \setminus K) \leq 2\varepsilon$. This shows $\mu_1 + \mu_2 \in M_r(X)$.

The argument above also reveals that "$\mu \in M_r(X) \Rightarrow c\mu \in M_r(X)$". Hence $M_r(X)$ is a subspace of $M(X)$. It remains to show that it is a closed subspace.

Let $\{\mu_n\} \subseteq M_r(X)$, $\mu \in M(X)$ be such that $\|\mu_n - \mu\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and $E \in \mathcal{B}$. There exists N such that $\|\mu_N - \mu\| \leq \varepsilon$. As $\mu_N \in M_r(X)$ and $|\mu_N|$ is a finite measure, there exists open set $G \supseteq E$ and compact set $K \subseteq E$ such that $|\mu_N|(G \setminus E) \leq \varepsilon$ and $|\mu_N|(E \setminus K) \leq \varepsilon$. Therefore,

$$\begin{aligned} |\mu|(G \setminus E) = |\mu - \mu_N + \mu_N|(G \setminus E) &\leq |\mu - \mu_N|(G \setminus E) + |\mu_N|(G \setminus E) \\ &\leq |\mu - \mu_N|(X) + \varepsilon \\ &= \|\mu - \mu_N\| + \varepsilon \leq 2\varepsilon, \end{aligned}$$

and similarly $|\mu|(E \setminus K) \leq 2\varepsilon$. We conclude that $\mu \in M_r(X)$.

(b) It is because $M_r(X)$ is a subspace and

$$\mu^+ = \frac{1}{2}|\mu| + \frac{1}{2}\mu, \quad \mu^- = \frac{1}{2}|\mu| - \frac{1}{2}\mu.$$

(c) We want to show that $\lambda \in M_r(X)$, where $\lambda(E) := \int_E f \, d|\mu|$.

Let $\varepsilon > 0$ and $E \in \mathcal{B}$. By lecture notes proposition 5.3, we have $|\lambda|(E) = \int_E |f| \, d|\mu|$. By Hw2 Q10, there exists $\delta > 0$ such that

$$\int_A |f| \, d|\mu| < \varepsilon \quad \text{whenever } |\mu|(A) < \delta.$$

Since $\mu \in M_r(X)$ and $|\mu|$ is a finite measure, there exists open set $G \supseteq E$ and compact set $K \subseteq E$ such that $|\mu|(G \setminus E) < \delta$ and $|\mu|(E \setminus K) < \delta$. Consequently

$$|\lambda|(G \setminus E) = \int_{G \setminus E} |f| \, d|\mu| < \varepsilon,$$

and similarly $|\lambda|(E \setminus K) < \varepsilon$. This was to be demonstrated.

11. At last, the TA of this course would like to express his acknowledgement. Firstly, I am very grateful that I can inherit the homework solutions. They are illuminating and save my time. Secondly, thank you for making your files more user-friendly to me, so that the TA work is less exhausting than it could be. Not only your mathematics but also your empathy demonstrates how great you are.