Lecture 1

1 What is ergodic theory?

Ergodic theory studies the asymptotic behaviour of measure preserving transformations on a measure space.

Let $(X, \mathscr{F}, \mu)$ be a probability space. A map $T : X \to X$ is called measurable if $T^{-1}A \in \mathscr{F}$ for each $A \in \mathscr{F}$. Furthermore, say T preserves μ, if $\mu(T^{-1}A) = \mu(A)$ for each $A \in \mathscr{F}$.

Ergodic theory originates from the study the statistical mechanics, to give a very rough description, let us consider a simple model as follows.

Imagine a vessel filled with gas of k molecules in total. Assume the masses and forces between them are complete known. At each moment, the position $\overrightarrow{q}$ and the velocity $\overrightarrow{v}$ of a molecule are both given by three coordinates, so the state of the gas is described by $6k$ coordinates, i.e. a point in $\mathbb{R}^{6k}$. We use $x_0 = \big((\overrightarrow{q_i}, \overrightarrow{v_i})\big)_{i=1}^{k}$ to denote the state of the gas at time 0, let x_t denote the state of gas at time t ($t \in \mathbb{R}$). The collection of all states is a subset of $\mathbb{R}^{6k}$, called the phase space.

Now consider the transformation $T_t : x_0 \to x_t$ ($t \in \mathbb{R}$).

In classical mechanics, given the initial state x_0, x_t is determined by solving differential equations according to the Laws of Motion. If the solution curve is unique, we have $T_{t+s} = T_t \circ T_s$, with this property, $\{T_t\}_{t\in\mathbb{R}}$ is called a flow.

$\{x_t\}_{t\in\mathbb{R}}$ is called a trajactory. A central question in classical mechanics is that: given x_0, what is the behaviour of x_t as $t \to \infty$? In other words, the behaviour of a single orbit is of interest. However, when k is huge(which is often the case), too many differential equations will be involved, consequently, solving them becomes impossible. To overcome this disadvantage, one of the founders of statistical mechanics named Gibbs suggested that rather than focusing on a single orbit, one should study subsets of the phase space using methods probabilistic or statistical in nature.

Gibbs: One should consider a subset E of phase space (an ensemble of state).

Question: Given E, what is the probability that the state of the system will be contained in E at time t?

Liouville proved there do exists T_t invariant measure on the phase space.

Liouville Theorem: There exists a “smooth” volume measure μ on the phase space ($\subset \mathbb{R}^{6k}$), such that T_t preserve μ for each $t \in \mathbb{R}$. Here “smooth” means μ is absolutely continuous w.r.t the Lebesgue measure.

To study the behaviour of T_t, we may consider a discrete model. Fix a $t_0 > 0$, denote $T = T_{t_0}$, then $T_{nt_0} = T_{t_0} \circ T_{t_0} \circ \cdots \circ T_{t_0}$. It is reasonable to believe the two flows $\{T_t\}_{t\in\mathbb{R}}$ and $\{T_n\}_{n\in\mathbb{Z}}$ have similar asymptotic behaviour as $t \to \infty$ and $n \to \infty$.

2 Topological dynamic systems and recurrence

Let (X, d) be a compact metric space. Let $T : X \to X$ be a continuous mapping, then (X, T) is called a topological dynamic system (TDS).

2.1 Examples

1.(doubling map)

Let $X = \mathbb{R}/\mathbb{Z}$ (the unit interval with 0 and 1 identified), define $T : X \to X$ by $Tx = 2x(\text{mod}1)$.

2.(rotation on the circle)

Let $X = \mathbb{R}/\mathbb{Z}$, let $\alpha \in (0, 1)$, define $T : X \to X$ by $Tx = x + \alpha(\text{mod}1)$.

3.(one-sided shift map)

Let $k \in \mathbb{N}, k > 1$, let $X = \{x = (x_n)_{n=1}^{\infty} : x_n \in \{1, 2, \cdots, k\}\}$. Define $d : X \times X \to [0, \infty)$ by

$$d\big((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\big) = \begin{cases} 2^{-\inf\{k:x_k \neq y_k\}} & x \neq y \\ 0 & x = y \end{cases}.$$

Define $T : X \to X$ by $T\big((x_n)_{n=1}^{\infty}\big) = \big(x_{n+1}\big)_{n=1}^{\infty}$.

A direct check shows d is a metric. For any sequence in X, there is a subsequence such that all elements in the subsequence have the same first coordinate, from this subsequence, we get a again a subsequence in which all elements have the same first two coordinates, inductively, we get a subsequence converges to a point in X, i.e. X is sequential compact, hence compact. T is clearly continuous.

4.(two-sided shift map)

Let $k \in \mathbb{N}, k > 1$, let $X = \{(x_n)_{n\in\mathbb{Z}} : x_n \in \{1, 2, \cdots, k\}\}$. Define $d : X \times X \to [0, \infty)$ by

$$d\big((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\big) = \begin{cases} 2^{-\inf\{|k|:x_k \neq y_k\}} & x \neq y \\ 0 & x = y \end{cases}.$$

Define $T : X \to X$ by $T\big((x_n)_{n=1}^{\infty}\big) = \big(x_{n+1}\big)_{n=1}^{\infty}$.

A similar argument of the previous example shows (X, T) is a TDS, note T in this case is a homeomorphism.

2.2 Recurrence

Definition 2.1. *x is said to be a periodic point of X if $T^n x = x$ for some $n \geq 1$.*

Examples: 1. On the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, define $Tx = x + 1/3$, then every point on $\mathbb{T}$ is periodic with period 3.

2. On $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, let $Tx = x+\alpha(\bmod 1)$, if α is irrational, there is no periodic point on $\mathbb{T}$.

Definition 2.2. *Let (X, T) be a TDS, say $x \in X$ is a recurrent point, if for any neighborhood $U \ni x$, there exists $n \geq 1$, such that $T^n x \in U$.*

Equivalently, x is said to be recurrent, if there exists a sequence of positive integers $\{n_i\}_{i=1}^{\infty}$, such that $T^{n_i} \to x$, as $i \to \infty$.

Unlike period point, recurrent point always exists in a TDS.

Theorem 2.1 (Birkhoff Recurrence Theorem)**.** *Let (X, T) be a TDS, then X has at least one recurrent point.*

The proof of this theorem needs an application of Zorn's lemma.

Let $(X, \leq)$ be a partially order set. A nonempty subset $\mathcal{C}$ of X is said to be a totally ordered chain, if for each pair $a, b \in \mathcal{C}$, either $a \leq b$ or $b \leq a$.

Zorn's lemma: If every chain $\mathcal{C}$ in $(X, \leq)$ has a lower bound, then X has a minimal element, that is there exists some $x \in X$ such that $y \leq x$ implies $y = x$.

Proof of Theorem 2.1 . Let $\mathcal{F} = \{Y \subset X \text{ nonemty and closed}, TY \subset Y\}$, note $\mathcal{F} \neq \emptyset$ since $X \in \mathcal{F}$. $\mathcal{F}$ is a partially ordered set under inclusion. Let $\mathcal{C} \subset \mathcal{F}$ be a totally ordered chain, since X is compact and $\mathcal{C}$ satisfies the finite intersection property, let $Y_0 = \cap_{Y \in \mathcal{C}} Y$, then Y_0 is nonempty and closed. Since $TY_0 \subset Y_0$ and $Y_0 \subset Y$ for every $Y \in \mathcal{C}$, $Y_0 \in \mathcal{F}$ and is a lower bound for $\mathcal{C}$. Then by Zorn's lemma, $\mathcal{F}$ has a minimal element, say Y_1, $Y_1 \neq \emptyset$ and closed, $TY_1 \subset Y_1$. Let $x \in Y_1$, we claim x is recurrent. Let $Q(x) = \overline{\{T^n x : n \geq 1\}}$, then $TQ(x) \subset Q(x)$ and $Q(x) \subset Y_1$, since Y_1 is a minimal element in $\mathcal{F}$, $Q(x) = Y$, in particular $x \in Q(x)$, which implies x is recurrent. □

Example: Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $Tx = x + \alpha(\text{mod}1)$, the every point $x \in \mathbb{T}$ is recurrent. Infact, by Birkhoff recurrence theorem, there is some $x_0 \in \mathbb{T}$ that is recurrent, hence there exists $\{n_i\}_{i=1}^{\infty}$, such that $T^{n_i} x_0 \to x_0$, note $T^{n_i} x_0 \to x_0 \Leftrightarrow n_i \alpha(\text{mod}1) \to 0 \Leftrightarrow x + n_i \alpha(\text{mod}1) \to 0, \forall x \in \mathbb{T} \Rightarrow x \text{ is recurrent}, \forall x \in \mathbb{T}$.

This example can be generalized to general compact groups (need not be abelian).

Definition 2.3 (Kronecker system)**.** *Let K be a compact group(may not be abelian), let $a \in K$, define $T : X \to X$ by $Tx = ax$(left multiplication).*

Proposition 2.1. *Let (K, T) be a Kronecker system, then every $x \in K$ is recurrent.*

Proof. By Birkhoff theorem, there exists some $x_0 \in K$ that is recurrent. Hence there exists $\{n_i\}_{i=1}^{\infty}$ such that $a^{n_i} x_0 \to x_0$, multiply both sides by x_0^{-1}, we get $a^{n_1} x_0 x_0^{-1} \to x_0 x_0^{-1} = e$(the identity element of K) $\Rightarrow a^{n_i} \to e \Rightarrow a^{n_i} x \to x, \forall x \in K \Rightarrow x$ is recurrent, $\forall x \in K$. □

Example (Higher dimensional torus, e.g. $\mathbb{T}^2$).

Let $\alpha_1, \alpha_2 \in (0,1)$, $T(x,y) := (x + \alpha_1(\text{mod}1), y + \alpha_2(\text{mod}1))$, according to Proposition 2.1, every point on $\mathbb{T}^2$ is recurrent.

2.3 Factors and extensions

Definition 2.4. *(X,T) and (Y,S) are two TDSs. Say (Y,S) is a factor of (X,T) if there exists $\pi : X \to Y$ continuous and surjective, such that the following diagram commutes,*

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{S} & Y \end{array}$$

that is $\pi \circ T = S \circ \pi$. X is called an extension of Y.

For instance, let $\Sigma = \{0,1\}^{\mathbb{N}}$ be a one-sided shift space. Let σ be the shift map. Define $\pi : \Sigma \to [0,1]$ by $\pi\big((x_n)_{n=1}^{\infty}\big) = \sum_{n=1}^{\infty} x_n 2^{-n}$, it's easy to see the following diagram commutes.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ [0,1] & \xrightarrow{2x(\text{ mod } 1)} & [0,1] \end{array}$$

Lemma 2.2. *Let (Y,S) be a factor of (X,T) with factor map $\pi : X \to Y$. If x is recurrent w.r.t T, then πx is recurrent w.r.t S.*

Proof. Let $x \in X$ be recurrent w.r.t T, then there exists $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}_+$, such that $T^{n_i}x \to x$, since π is continuous, $\pi(T^{n_i}x) \to \pi x$. Since $\pi \circ T^n = S^n \circ \pi$ for each $n \in \mathbb{N}_+$, we have $S^{n_i}(\pi x) \to \pi x$, therefore πx is recurrent w.r.t S. $\square$