TA's solution to 3093 assignment 6

Ch4, Ex4. (4 marks)[1]

Before we start, let's consider an example:

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be defined by

$$\gamma(t) = \begin{cases} (\cos 2t, \sin 2t) & \text{if } t \in [0, \pi) \\ (2 - \cos 2t, \sin 2t) & \text{if } t \in [\pi, 2\pi]. \end{cases}$$

This curve Γ looks like the "∞" symbol. As

$$\gamma'(t) = \begin{cases} (-2\sin 2t, 2\cos 2t) & \text{if } t \in [0, \pi) \\ (2\sin 2t, 2\cos 2t) & \text{if } t \in (\pi, 2\pi], \end{cases}$$

and

$$\lim_{h\to 0^+} \frac{\gamma_1(\pi + h) - \gamma_1(\pi)}{h} = \lim_{h\to 0^+} \frac{2 - \cos(2\pi + 2h) - 1}{h} = 0 \qquad \text{(by L'Hospital's Rule)}$$
$$= \lim_{h\to 0^-} \frac{\cos(2\pi + 2h) - 1}{h} = \lim_{h\to 0^-} \frac{\gamma_1(\pi + h) - \gamma_1(\pi)}{h},$$

we see that $\gamma \in \mathcal{C}^1$, and $|\gamma'(t)| \neq 0$ for all $t \in [0, 2\pi]$. Note that

$$\int_0^{2\pi} (\gamma_1\gamma_2' - \gamma_2\gamma_1') = \int_0^{\pi} (2\cos^2 2t + 2\sin^2 2t)dt + \int_{\pi}^{2\pi} (4\cos 2t - 2\cos^2 2t - 2\sin^2 2t)dt$$
$$= \int_{\pi}^{2\pi} (4\cos 2t)dt = 0.$$

To have an understanding of the isoperimetric inequality when the curve is not simple, please refer to the following excerpts[2]:

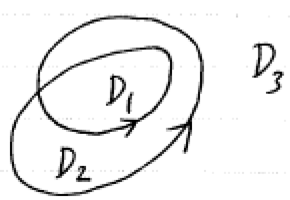
> the isoperimetric problem is : Among all closed, simple curves with fixed length 2π, only the unit circle has the maximal area π.

[1]The material and idea presented here are from the 2007-08 MAT3090 class by Prof. Chou Kai Seng (and his TA?).
[2]They are from lecture notes 8 of the class mentioned in the first footnote.

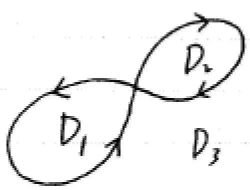
We observe the above proof does not use the fact that γ is simple. this condition is used to make sure that the area A has a geometric meaning. Hence, for any γ in positive direction we define

$$A = \frac{1}{2}\int_{\gamma}(x\,dy - y\,dx)$$

the isoperimetric problem is correct. It is interesting to assign a meaning to this integral when the curve has self-intersection. The answer turns out depending on the notion of the winding number. γ divides $\mathbb{R}^2$ into finitely many disjoint domains $D_1, D_2, \ldots, D_N$ where D_N is usually assigned for the outer unbdd domain. For each D_j we can assign it with an integer $n_j \in \mathbb{Z}$ called the winding number of the domain.



$n_1 = 2$
$n_2 = 1$
$n_3 = 0$



$n_1 = 1$
$n_2 = -1$
$n_3 = 0$

then one can show that

$$\frac{1}{2}\int_{\gamma}(x\,dy - y\,dx) = \sum_{1}^{N} n_k |D_k|,$$

(γ: in positive or negative direction.) ($|D_k|$: the enclosed (geometric) area of D_k.)

this integral is better interpreted as the signed area. A stronger notion of area is the weighted area Date

$$A_W = \sum_{k=1}^{N} |n_k|\,|D_k| \,.$$

By the homogenity of the Euclidean space, the isoperimetric ~~inequalit~~ problem is equivalent to the isoperimetric inequality:

$$L^2 \geqslant 4\pi A$$

for any C^1-curve where L and A are defined in (1) and (2) γ (Exercise)

"=" holds $\iff$ γ is a circle.

The Rado inequality (1935) gives a better result

$$L^2 \geqslant 4\pi A_W \,.$$

More, the Banchoff-Pohl (1971) inequality asserts

$$L^2 \geqslant 4\pi \sum_{k=1}^{N} n_k^2 |D_k| \quad !$$

Let's come back to the question now. We shall show the equivalence of the following two statements:

(S1) Given any $\gamma : [a,b] \to \mathbb{R}^2$, where $\gamma \in \mathcal{C}^1$, $|\gamma'(t)| \neq 0$ on $[a,b]$, and $\gamma(a) = \gamma(b)$, writing $\gamma(t) = (x(t), y(t))$ we have

$$\alpha_\gamma \leq \frac{\beta_\gamma^2}{4\pi},$$

where

$$\alpha_\gamma := \left| \int_a^b x'y \right| \quad \text{and} \quad \beta_\gamma := \int_a^b \sqrt{(x')^2 + (y')^2}.$$

(S2) Given any 2π-periodic function $f \in \mathcal{C}^1$ with $\int_0^{2\pi} f = 0$, we have

$$\int_0^{2\pi} |f|^2 \leq \int_0^{2\pi} |f'|^2 .$$

(S1)⇒(S2):

Since $|f|^2 = \Re(f)^2 + \Im(f)^2$ and $|f'|^2 = (\Re(f)')^2 + (\Im(f)')^2$, we can assume that f is a real-valued function rather than a probably complex-valued function. If $f' \equiv 0$ on $[0, 2\pi]$, then f is a constant function. By $\int_0^{2\pi} f = 0$, we see that $f \equiv 0$ as well, so the desired inequality holds plainly. Therefore, we can assume that f' is not the zero function on $[0, 2\pi]$.

Given $\varepsilon > 0$, for all large $N \in \mathbb{N}$ we have $\|f - S_N(f)\|_2 \leq \varepsilon$ and $\|f' - S_N(f')\|_2 \leq \varepsilon$.[3] Note that $S_N(f')(\theta) = \frac{d}{d\theta} S_N(f)(\theta)$[4]. Since f is real-valued, we have $\widehat{f}(-n) = \overline{\widehat{f}(n)}$, whence both $S_N(f)$ and $S_N(f')$ are real-valued functions. Notice also that since f' is not the zero function, $S_N(f)$ and $S_N(f')$ are not the zero trigonometric polynomial for all large N[5].

Fix a large N satisfying the aforementioned requirements. By the substitution $x = e^{i\theta}$ and the fundamental theorem of algebra, we see that the equation $S_N(f')(\theta) = 0$ can only have finitely many solutions for $\theta \in [0, 2\pi]$. This implies the equation $S_N(f)^2(\theta) + S_N(f')^2(\theta) = 0$ can only have finitely many solutions for $\theta \in [0, 2\pi]$.

Here we make an assumption: there exists a 2π-periodic real-valued function σ satisfying the following properties:

- $\sigma \in \mathcal{C}^\infty$. i.e. σ is infinitely differentiable.
- $\int_0^{2\pi} \sigma = 0$.
- $\|\sigma\|_2 < \varepsilon$ and $\|\sigma'\|_2 < \varepsilon$.
- $(S_N(f) + \sigma)^2 + (S_N(f') + \sigma')^2 > 0$ on $[0, 2\pi]$.

We shall justify this assumption later.

Under this assumption, writing $S_{N,\sigma} := S_N(f) + \sigma$, we consider $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\gamma(t) := \left(-\int_0^t S_{N,\sigma}(\theta) d\theta, \quad S_{N,\sigma}(t) \right).$$

We have $\gamma \in \mathcal{C}^1$, and $|\gamma'(t)|^2 \neq 0$. Since $\int_0^{2\pi} f = 0$, the constant coefficient of $S_N(f)$ is zero, whence $\gamma(0) = \gamma(2\pi) = (0,\ S_{N,\sigma}(0))$. We are allowed to use (S1) now.

We get

$$\left| \int_0^{2\pi} (S_{N,\sigma})^2(\theta) d\theta \right| \leq \frac{1}{4\pi} \left(\int_0^{2\pi} \sqrt{(S_{N,\sigma})^2(\theta) + (S'_{N,\sigma})^2(\theta)} d\theta \right)^2 .$$

It follows that

$$\begin{aligned} \int_0^{2\pi} (S_{N,\sigma})^2(\theta) d\theta &\leq \frac{1}{4\pi} \left(\int_0^{2\pi} ((S_{N,\sigma})^2 + (S'_{N,\sigma})^2) \right) \left(\int_0^{2\pi} 1 \right) \quad \text{(Cauchy-Schwarz inequality)} \\ &= \frac{1}{2} \left(\int_0^{2\pi} (S_{N,\sigma})^2(\theta) d\theta + \int_0^{2\pi} (S'_{N,\sigma})^2(\theta) d\theta \right). \end{aligned}$$

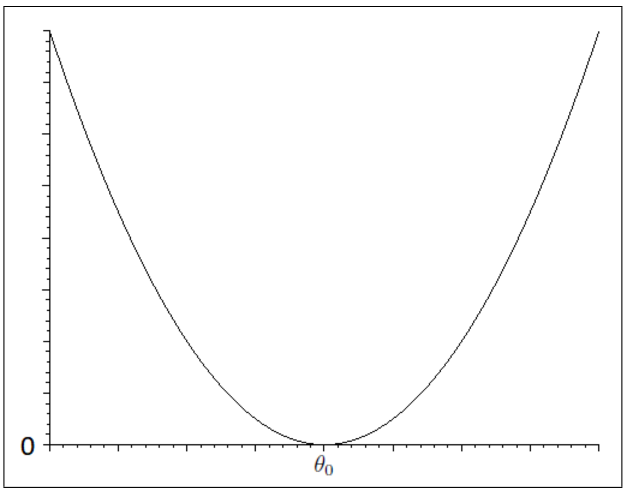
[3]This is by textbook Ch3 Theorem 1.1.
[4]E.g. by textbook Ch2 p.43
[5]This is by textbook Ch2 Corollary 2.2.

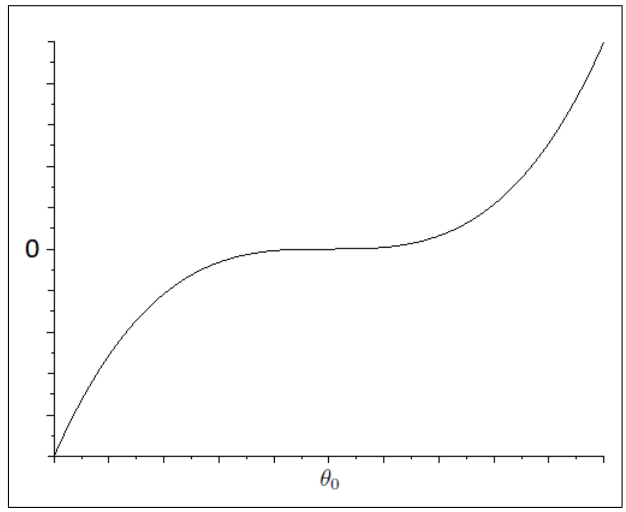
Hence $\|S_N(f)+\sigma\|_2 = \|S_{N,\sigma}\|_2 \le \left\|S'_{N,\sigma}\right\|_2 = \|S_N(f')+\sigma'\|_2$. Consequently,

$$\begin{aligned}\|f\|_2 &\le \|f - S_N(f)\|_2 + \|S_N(f)\|_2 \le \varepsilon + \|S_N(f)\|_2 \\ &\le \varepsilon + \|S_N(f)+\sigma\|_2 + \|\sigma\|_2 \\ &\le \varepsilon + \|S_N(f')+\sigma'\|_2 + \|\sigma\|_2 \\ &\le \varepsilon + \|S_N(f') - f'\|_2 + \|f'\|_2 + \|\sigma'\|_2 + \|\sigma\|_2 \\ &\le 4\varepsilon + \|f'\|_2 .\end{aligned}$$

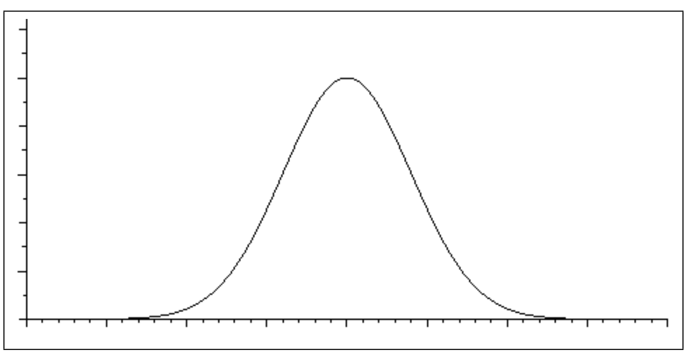
Since $\varepsilon > 0$ can be arbitrarily small, the result follows.

It remains to justify such σ exists. Since we have already done so much, let's try to do it casually. The idea is to *perturb the function $S_N(f)$ a little bit*[6] at each small neighborhood of θ_0 when $S_N(f)^2(\theta_0) + S_N(f')^2(\theta_0) = 0$. We may only need to consider two situations:



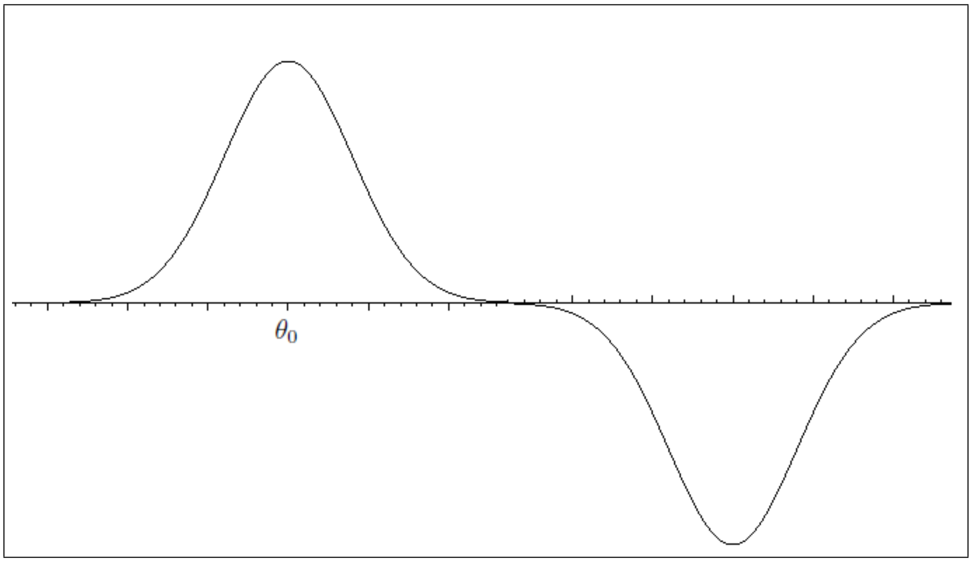


Suppose we have an infinitely differentiable "bump function" which vanishes outside a bounded interval and looks like the following:

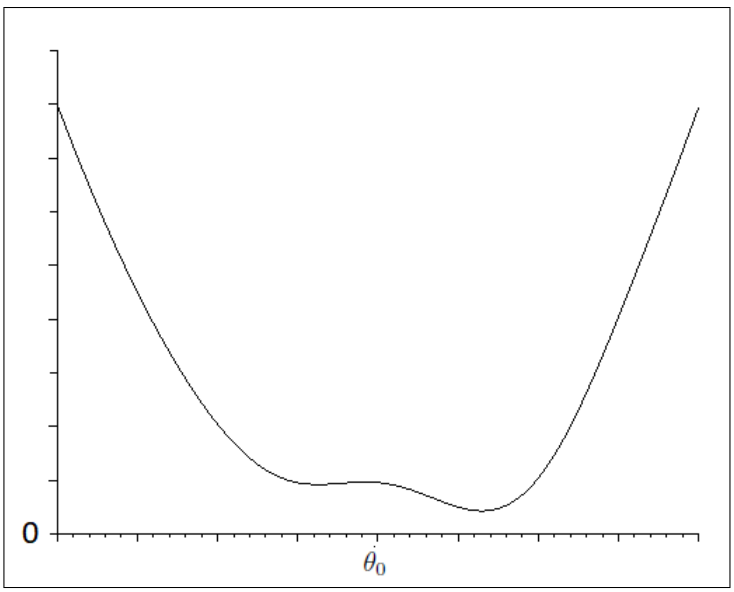


Then we can make copies of it and combine them through translations and scalar multiplications. Therefore, in the first situation, we may use the following σ_1:

[6]This is the words from the suggested solutions to exercise 8 of the class mentioned in the first footnote.

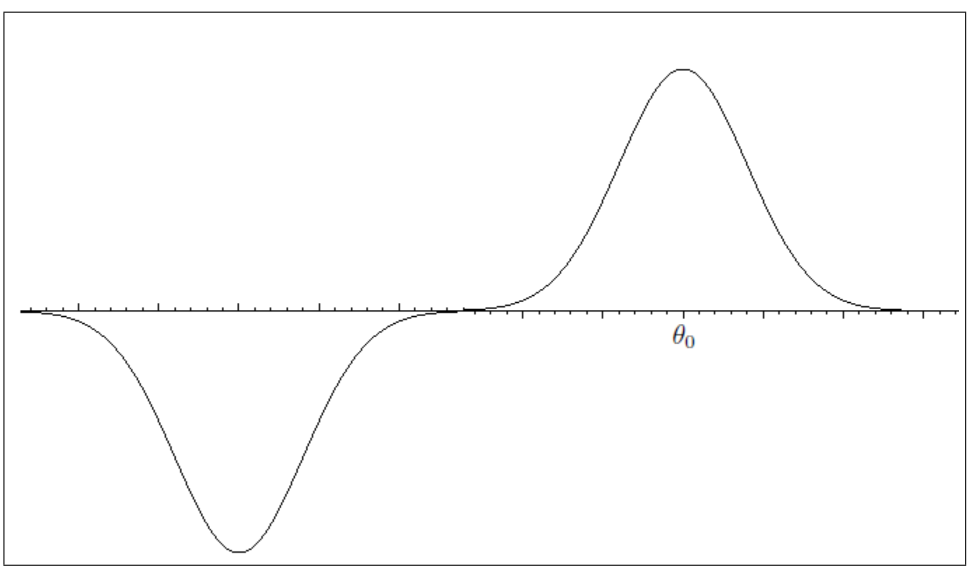


Then near θ_0, $S_N(f) + \sigma_1$ may be like:

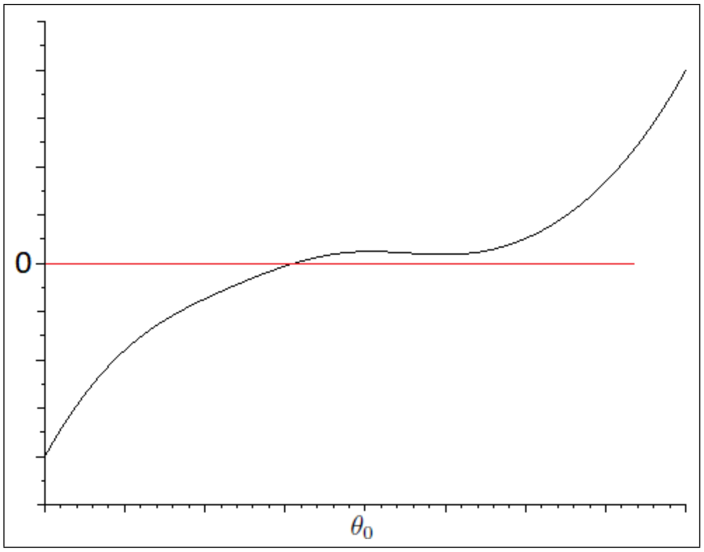


so that $(S_N(f) + \sigma_1)(\theta) > 0$ near θ_0.

In the second situation, we may use the following σ_2:



Near θ_0, $S_N(f) + \sigma_2$ may be like:



so that when $(S_N(f) + \sigma_2)(\theta_1) = 0$, we have $(S_N(f') + \sigma_2')(\theta_1) > 0$.

The existence of such bump functions may be guaranteed by considering the following function: given $\delta > 0$, define

$$g_\delta(x) := \begin{cases} e^{-\frac{1}{\delta^2 - (x-\theta_0)^2}} & \text{if } |x - \theta_0| < \delta \\ 0 & \text{otherwise.}^7 \end{cases}$$

(S2)⇒(S1):

Define $h : [a, b] \to [0, \beta_\gamma]$ by $h(s) := \int_a^s |\gamma'(t)|\, dt$. Since $|\gamma'| \neq 0$, we can follow the idea of Ch4 Ex1 and consider $\rho : [0, \beta_\gamma] \to \mathbb{R}^2$ defined by $\rho = \gamma \circ h^{-1}$. Writing $\rho(s) = (u(s), v(s))$, it satisfies

$$(u')^2 + (v')^2 \equiv 1, \quad \text{and} \quad \left| \int_0^{\beta_\gamma} u'(s)v(s)ds \right| = \left| \int_a^b x'(t)y(t)dt \right| \quad \text{(by the substitution } s = h(t)\text{)}.$$

As $|\gamma'| \neq 0$, we have $\beta_\gamma > 0$. Define

$$J := \begin{cases} 1 & \text{if } \int_0^{\beta_\gamma} u'(s)v(s)ds \geq 0 \\ -1 & \text{otherwise,} \end{cases}$$

$$V(s) := v(s) - \frac{1}{\beta_\gamma} \int_0^{\beta_\gamma} v(\xi)d\xi,$$

and

$$c := \frac{2\pi}{\beta_\gamma}.$$

[7] C.f. this stackexchange post and textbook Ch5 Ex4.

Noting that $(u')^2 + (V')^2 \equiv 1$ and $J^2 = 1$, we have

$$\begin{aligned}
\frac{\beta_\gamma^2}{4\pi} &= \frac{\beta_\gamma}{4\pi}\int_0^{\beta_\gamma}((u')^2+(V')^2) = \frac{\beta_\gamma}{4\pi}\int_0^{\beta_\gamma}\left[(u'-JcV)^2+((V')^2-c^2V^2)+2Jcu'V\right] \\
&\geq \frac{\beta_\gamma}{4\pi}\int_0^{\beta_\gamma}\left[(V')^2-c^2V^2\right]+\frac{\beta_\gamma 2c}{4\pi}\cdot\left(J\int_0^{\beta_\gamma}u'V\right) \\
&= \frac{\beta_\gamma}{4\pi}\int_0^{\beta_\gamma}\left[(V')^2-c^2V^2\right]+J\int_0^{\beta_\gamma}u'V.
\end{aligned}$$

Since $x(b) = x(a)$, we have $\int_0^{\beta_\gamma} u'A = A(u(\beta_\gamma) - u(0)) = 0$ for any constant A. Therefore,

$$J\int_0^{\beta_\gamma}u'V = \left|\int_0^{\beta_\gamma}u'v\right| = \left|\int_a^b x'(t)y(t)dt\right|.$$

It remains to show that

$$\int_0^{\beta_\gamma}\left[(V')^2-c^2V^2\right] \geq 0.$$

Define $f : [0, 2\pi] \to \mathbb{R}$ by

$$f(\theta) := V(\frac{\beta_\gamma}{2\pi}\theta).$$

Then

$$\int_0^{2\pi}(f)^2 = \int_0^{2\pi}\left(V(\frac{\beta_\gamma}{2\pi}\theta)\right)^2 d\theta = \left(\frac{2\pi}{\beta_\gamma}\right)\int_0^{\beta_\gamma}(V(\xi))^2\,d\xi = \int_0^{\beta_\gamma}cV^2,$$

and

$$\int_0^{2\pi}(f')^2 = \left(\frac{\beta_\gamma}{2\pi}\right)^2\int_0^{2\pi}\left(V'(\frac{\beta_\gamma}{2\pi}\theta)\right)^2 d\theta = \left(\frac{\beta_\gamma}{2\pi}\right)\int_0^{\beta_\gamma}(V'(\xi))^2\,d\xi = \frac{1}{c}\int_0^{\beta_\gamma}(V')^2.$$

Since $f(0) = f(2\pi)$, we can extend f to be a 2π-periodic function. We have $f \in \mathcal{C}^1$ and

$$\int_0^{2\pi} f = \frac{2\pi}{\beta_\gamma}\int_0^{\beta_\gamma}V(s)ds = \frac{2\pi}{\beta_\gamma}\left[\int_0^{\beta_\gamma}v(s)ds - \int_0^{\beta_\gamma}v(s)ds\right] = 0.$$

By (S2), we have $\int_0^{2\pi} f^2 \leq \int_0^{2\pi}(f')^2$. Done[8].

Ex5. (3 marks) Most students have no problem about this question. A solution may be[9]

> Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then γ_n is the fractional part of α^n. Now we let $U_n = \alpha^n + \beta^n$, so $U_0 = 2$ and $U_1 = 1$. Now since α and β are the two roots of the quadratic equation $x^2 = x + 1$, for any $r \geq 1$ we must have $U_{r+1} = \alpha^{r+1} + \beta^{r+1} = \alpha^r + \alpha^{r-1} + \beta^r + \beta^{r-1} = U_r + U_{r-1}$. Therefore U_n is an integer for any n. Now we notice that $|\beta| < 1$, so for sufficiently large n, $|\beta^n| < 1/3$. Therefore, since $\alpha^n = U_n - \beta^n$, $\alpha^n \in (U_n - 1/3, U_n + 1/3)$, implying $\gamma_n \notin (1/3, 2/3)$. Hence $\#\{1 \leq n \leq N : \gamma_n \in (1/3, 2/3)\}$ is a constant for sufficiently large N, so $\lim_{N\to\infty}\frac{\#\{1 \leq n \leq N : \gamma_n \in (1/3,2/3)\}}{N} = 0$. Hence $\{\gamma_n\}_{n=1}^{\infty}$ is not equidistributed in $[0, 1]$.

[8] In view of load management, let's skip the "equality holds if and only if" part.

[9] A student provides this solution.

Let's make a remark. Suppose $\theta_0 \in (1, \infty)$ satisfies the following properties:

- There exist $\theta_1, \dots, \theta_d \in \mathbb{C}$ such that $\theta_0^n + \theta_1^n + \cdots + \theta_d^n \in \mathbb{Z}$ for all $n \in \mathbb{N}$;
- $|\theta_i| < 1 \ \forall 1 \le i \le d$.

Then by the same argument as above, we see that the fractional part of θ_0^n is not equidistributed in $[0, 1]$. The Pisot numbers, which includes the golden ratio $\frac{1+\sqrt{5}}{2}$, are examples of such θ_0.

Ex10. (3 marks)

(a). A solution to this part may be[10]

By Weyl's criterion, for all integers $k \neq 0$ we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0$, so we see

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k(x+\xi_n)} = \lim_{N\to\infty} \frac{e^{2\pi i k x}}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0,$$

and the limit is uniform to all x. Therefore for any trigonometric polynomial $P(x) = \sum_{k=-K}^{K} c_k e^{2\pi i k x}$ with $\int_0^1 P(x)dx = 0$, $c_0 = 0$, we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} P(x+\xi_n) = 0$

Now for a fixed continuous f with $\int_0^1 f(x)dx = 0$, for any $\varepsilon > 0$ there exists some trigonometric polynomial P such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [0, 1]$. Then we note that $\left|\int_0^1 P(x)dx\right| = \left|\int_0^1 (P(x) - f(x))dx\right| \le \int_0^1 |P(x) - f(x)|dx \le \varepsilon$, so we can let $r = \int_0^1 P(x)dx$ and obtain $|r| \le \varepsilon$. Denote $Q(x) = P(x) - r$, so $Q(x)$ is a trigonometric polynomial with $\int_0^1 Q(x) = 0$. So we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} Q(x+\xi_n) = 0$, so that for sufficiently large N, $\left|\frac{1}{N}\sum_{n=1}^{N} Q(x+\xi_n)\right| < \varepsilon$. Now

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right| \\
&\le \left|\frac{1}{N}\sum_{n=1}^{N} (f(x+\xi_n) - P(x+\xi_n))\right| + \left|\frac{1}{N}\sum_{n=1}^{N} (P(x+\xi_n) - Q(x+\xi_n))\right| + \left|\frac{1}{N}\sum_{n=1}^{N} Q(x+\xi_n)\right| \\
&< \frac{1}{N}\sum_{n=1}^{N} |f(x+\xi_n) - P(x+\xi_n)| + \frac{1}{N}\sum_{n=1}^{N} |r| + \varepsilon \\
&\le \frac{N\varepsilon}{N} + \frac{N|r|}{N} + \varepsilon \le 3\varepsilon.
\end{aligned}$$

Therefore we must have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n) = 0$ as desired.

[10] This solution is adapted from a student's work.

(b).

Approach 1

Given $\varepsilon > 0$, we have $\|f - S_m(f)\|_2 < \varepsilon$ for some large m[11]. Since $\int_0^1 f = 0$, the constant term of $S_m(f)$ is zero. Therefore, $S_m(f)$ is a continuous function satisfying $\int_0^1 S_m(f) = 0$. By the result of part (a), we have

$$\lim_{n\to\infty} \frac{1}{N} \sum_1^N S_m(f)(x+\xi_n) = 0 \quad \text{uniformly in } x.$$

Consequently, there exists L s.t. for all $N \geq L$, for all $x \in [0,1]$, we have

$$\left| \frac{1}{N} \sum_1^N S_m(f)(x+\xi_n) \right| < \varepsilon,$$

whence

$$\begin{aligned}
&\int_0^1 \left| \frac{1}{N} \sum_1^N f(x+\xi_n) \right|^2 dx \\
&= \int_0^1 \left| \frac{1}{N} \sum_1^N [f(x+\xi_n) - S_m(f)(x+\xi_n)] + \frac{1}{N} \sum_1^N S_m(f)(x+\xi_n) \right|^2 \\
&\overset{12}{\leq} \int_0^1 2 \left| \frac{1}{N} \sum_1^N [f(x+\xi_n) - S_m(f)(x+\xi_n)] \right|^2 dx + \int_0^1 2 \left| \frac{1}{N} \sum_1^N S_m(f)(x+\xi_n) \right|^2 dx \\
&\leq 2 \left\| \frac{1}{N} \sum_1^N [f(x+\xi_n) - S_m(f)(x+\xi_n)] \right\|_2^2 + 2\varepsilon^2 \\
&\leq 2 \left(\frac{1}{N} \sum_1^N \|f - S_m(f)\|_2 \right)^2 + 2\varepsilon^2 \leq 4\varepsilon^2.
\end{aligned}$$

The result follows.

Approach 2[13]

Suppose f is integrable and $\int_0^1 f\mathrm{d}x = 0$. Let $g_n(x) = f(x+\xi_n)$. Then $\hat{g_n}(k) = \int_0^1 g_n(x)e^{-2\pi ikx}\mathrm{d}x = e^{2\pi ik\xi_n}\int_0^1 f(x+\xi_n)e^{-2\pi ik(x+\xi_n)}\mathrm{d}x = e^{2\pi ik\xi_n}\hat{f}(k)$ for $k \neq 0$, $\hat{g_n}(0) = \int_0^1 f(x+\xi_n)\mathrm{d}x = \int_0^1 f\mathrm{d}x = 0$.

Let $f_N(x) = \frac{1}{N}\sum_{n=1}^N f(x+\xi_n) = \frac{1}{N}\sum_{n=1}^N g_n(x)$. Then $\hat{f_N}(0) = \frac{1}{N}\sum_{n=1}^N \hat{g_n}(0) = 0$, $\forall k \neq 0$, $\hat{f_N}(k) = \frac{1}{N}\sum_{n=1}^N \hat{g_n}(k) = \hat{f}(k)\frac{1}{N}\sum_{n=1}^N e^{2\pi ik\xi_n} \xrightarrow{N\to\infty} 0$. Also, $|\hat{f_N}(k)| = |\hat{f}(k)| \cdot \frac{1}{N}|\sum_{n=1}^N e^{2\pi ik\xi_n}| \leq |\hat{f}(k)|$. Since f is Riemann integrable, f^2 is also Riemann integrable, and so $\infty > \int_0^1 |f|^2\mathrm{d}x = \sum_k |\hat{f}(k)|^2 \geq \sum_k |\hat{f_N}(k)|^2$. So by dominated convergence theorem, $\lim_N \int_0^1 \left|\frac{1}{N}\sum_{n=1}^N f(x+\xi_n)\right|^2 \mathrm{d}x = \lim_N \int_0^1 |f_N|^2\mathrm{d}x = \lim_N \sum_k |\hat{f_N}(k)|^2 = \sum_k \lim_N |\hat{f_N}(k)|^2 = 0$.

[11] This may be the reason why the authors give us a square in this question.
[12] By $|a+b|^2 \leq (|a|+|b|)^2 \leq 2|a|^2 + 2|b|^2$.
[13] A student provides this solution.

A more elementary argument for the highlighted part may be as follows. Fix a $\varepsilon > 0$. Since $\sum_{-\infty}^{\infty} \left|\widehat{f}(k)\right|^2 < \infty$, there exists K s.t. $\sum_{|k|\geq K} \left|\widehat{f}(k)\right|^2 < \varepsilon$. Then there exists L s.t. $\sum_{|k|\leq K} \left|\widehat{f_N}(k)\right|^2 < \varepsilon$ for all $N \geq L$. Consequently, for all $N \geq L$ we have

$$\sum_{-\infty}^{\infty} \left|\widehat{f_N}(k)\right|^2 = \sum_{|k|\leq K} \left|\widehat{f_N}(k)\right|^2 + \sum_{|k|>K} \left|\widehat{f_N}(k)\right|^2 \leq \sum_{|k|\leq K} \left|\widehat{f_N}(k)\right|^2 + \sum_{|k|>K} \left|\widehat{f}(k)\right|^2 \leq 2\varepsilon.$$

Approach 3[14]

Ex 10(b). For any $\epsilon > 0$ and any Riemann integrable functions f, by Lemma 3.2 in Chapter 2 of the book, there exists a continuous function g such that

$$\sup_{x\in[0,1]} |g(x)| \leq \sup_{x\in[0,1]} |f(x)| \text{ and } \int_0^1 |f(x) - g(x)|dx < \epsilon.$$

Define $h(x) = g(x) - \int g(x)dx$. Then h satisfies condition in (a), so that $\frac{1}{N}\sum_{n=1}^{N} h(x+\xi_n) \to 0$ uniformly in x. Hence, this means that for N large

$$\left|\frac{1}{N}\sum_{n=1}^{N} g(x+\xi_n) - \int_0^1 g(x)dx\right| < \epsilon \text{ uniformly in } x.$$

Let $M = \sup_{x\in[0,1]} |f(x)|$, note that $\int f(x)dx = 0$, we have

$$\begin{aligned}
\int_0^1 \left|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right|^2 dx \leq & M \int_0^1 \left|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right| dx \\
\leq & M \int_0^1 \left|\frac{1}{N}\sum_{n=1}^{N} (f(x+\xi_n) - g(x+\xi_n))\right| dx \\
& + M \int_0^1 \left|\frac{1}{N}\sum_{n=1}^{N} g(x+\xi_n) - \int_0^1 g(x)dx\right| dx \\
& + M \int_0^1 \left|\int_0^1 g(x)dx - \int_0^1 f(x)dx\right| dx \\
< & M \int_0^1 | f(x) - g(x)| dx + 2M\epsilon \\
< & 3M\epsilon.
\end{aligned}$$

This establishes the result.

We remark that

- We should also check if h is of period 1 before applying part (a).
- This approach makes no use of the square. The same argument works for any positive integer power.

[14] This solution is adapted from the work by former TAs.