TA's solution to 3093 assignment 5

Ch3, Ex11. (a). (It seems that you already have good solution from tutorial??)

(b). (2 marks)

As the tutorial note shows, the trick is to consider the function

$$G(t) := g(t) - \frac{1}{T}\int_0^T g(x)dx$$

so that we can apply the result of part(a).

As some students suggest, we may also do it as follows:

Although $\int_0^T g(x)dx$ may not be zero, we always have

$$\widehat{g'}(0) = \frac{1}{T}\int_0^T g'(x)dx = \frac{1}{T}\left(g(T) - g(0)\right) = 0,$$

and

$$\widehat{g'}(n) = \frac{1}{T}\int_0^T g'(x)e^{-inx2\pi/T}dx = \frac{in2\pi}{T}\widehat{g}(n).$$

Hence

$$\begin{aligned}
\left|\int_0^T \overline{f}g\right|^2 &= \left|T\sum_{-\infty}^{\infty}\overline{\widehat{f}(n)}\widehat{g}(n)\right|^2 \quad \text{(by Parseval identity)}\\
&= T^2\left|\sum_{\substack{-\infty<n<\infty\\ n\neq 0}}\overline{\widehat{f}(n)}\widehat{g}(n)\right|^2 \quad \text{(as } \widehat{f}(0)=0 \text{ by hypothesis)}\\
&\leq T^2\sum_{\substack{-\infty<n<\infty\\ n\neq 0}}\left|\widehat{f}(n)\right|^2\sum_{\substack{-\infty<n<\infty\\ n\neq 0}}\left|\widehat{g}(n)\right|^2 \quad \text{(Cauchy-Schwarz inequality)}\\
&\leq T^2\sum_{-\infty}^{\infty}\left|\widehat{f}(n)\right|^2\sum_{-\infty}^{\infty}|n|^2\left|\widehat{g}(n)\right|^2\\
&= T^2\sum_{-\infty}^{\infty}\left|\widehat{f}(n)\right|^2\sum_{-\infty}^{\infty}\left|\widehat{g'}(n)\right|^2\frac{T^2}{4\pi^2} = \left(\int_0^T |f|^2\right)\left(\int_0^T |g'|^2\right)\frac{T^2}{4\pi^2} \quad \text{(by Parseval identity).}
\end{aligned}$$

(c). / (Please refer to the tutorial note)

Ex15. (a). /

(b). (1 marks)

Please refer to the tutorial note for a solution. This question should not be difficult, so let's try something new. We introduce the Vinogradov notation*:

Given $f,\, g : X \to [0,\infty)$, we write $f \ll g$ if there exists a positive constant $C > 0$ s.t. for all $x \in X$, we have

$$f(x) \leq Cg(x).$$

For example,

*To me this notation is very useful. However, it is not a conventional notation in Mathematics (perhaps because the symbol used is somewhat misleading). Therefore, I suggest you mention the name "Vinogradov" whenever you use it.

- For all $x \in [1,\infty)$, we have $\dfrac{7x^4}{5x^3+2x+1} \ll x$;
- Given a fixed $\varepsilon_0 > 0$, we have $\log x \ll x^{\varepsilon_0}$ for all $x \in [1,\infty)$;
- $500 \ll 1$;
- $\sum_{n=1}^{\infty} \dfrac{1}{n^2} \ll 1$.

The first holds because $\dfrac{7x^4}{5x^3+2x+1} \leq \dfrac{35x^4+14x^2+7x}{5x^3+2x+1} = 7x$, so the implied constant C can be 7. The second holds because $\lim_{x\to\infty} \dfrac{\log x}{x^{\varepsilon_0}} = 0$ by L'Hospital's Rule. As $500 \leq 500 \cdot 1$, the implied constant for the third example can be 500. Finally, the forth holds because the series $\sum \dfrac{1}{n^2}$ converges.

We can also use subscripts to indicate the dependency of the implied constant. For example, we write $\log x \ll_\varepsilon x^\varepsilon$, meaning that for some constant $C_\varepsilon > 0$ depending on ε, we have $\log x \leq C_\varepsilon x^\varepsilon$ for all $x \in [1,\infty)$.

With this notation, a solution to this hw question may be as follows: Assuming

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]e^{-inx}dx,$$

we have

$$\left|\widehat{f}(n)\right| \ll \int_{-\pi}^{\pi} \left|f(x) - f(x+\frac{\pi}{n})\right| \ll \int_{-\pi}^{\pi} \left|\frac{\pi}{n}\right|^{\alpha} \ll_\alpha \frac{1}{|n|^\alpha}.$$

(c). With the Vinogradov notation, a solution may also be written as follows: We have

$$f(x+h) - f(x) = \sum_{\substack{0\leq k<\infty \\ 2^k \leq 1/|h|}} 2^{-k\alpha} e^{i2^k x}(e^{i2^k h} - 1) + \sum_{\substack{0\leq k<\infty \\ 2^k > 1/|h|}} 2^{-k\alpha} e^{i2^k x}(e^{i2^k h} - 1) := \Sigma_1 + \Sigma_2.$$

Since[†]

$$\left|e^{i2^k h} - 1\right| = \left|i\int_0^{2^k h} e^{it}dt\right| \leq \int_0^{2^k|h|} \left|e^{it}\right| dt = 2^k\left|h\right|,$$

we have

$$|\Sigma_1| \leq \sum_{\substack{0\leq k<\infty \\ 2^k \leq 1/|h|}} 2^{-k\alpha} 2^k |h| := |h| \frac{(2^{1-\alpha})^? - 1}{2^{1-\alpha} - 1}$$

$$\ll_\alpha |h|\left((2^{1-\alpha})^? - 1\right) \leq |h|\,(2^?)^{1-\alpha} \ll_\alpha |h| \left(\frac{1}{|h|}\right)^{1-\alpha} = |h|^\alpha.$$

As

$$|\Sigma_2| \ll \sum_{\substack{0\leq k<\infty \\ 2^k > 1/|h|}} 2^{-k\alpha} \ll_\alpha{}^{\ddagger} |h|^\alpha,$$

we have $|f(x+h) - f(x)| \leq |\Sigma_1| + |\Sigma_2| \ll_\alpha |h|^\alpha$. The result follows.

[†] The general form of this trick is that for $f : [a,b] \to \mathbb{C}$, $|f(b) - f(a)| = \left|\int_a^b f'(t)dt\right| \leq \int_a^b |f'(t)|\,dt \leq \sup|f'| \cdot |b-a|$. This trick is useful as we do not have mean value theorem for complex-valued functions.

[‡] It is because the sum is $:= \dfrac{2^{-?'\alpha}}{1-2^{-\alpha}} \ll_\alpha 2^{-?'\alpha}$, where $2^{?'} > 1/|h|$.

Ex16. (a). /

(b). (2 marks) /

(c). /

(d). With the Vinogradov notation, a solution may also be written as follows:
Assuming

$$\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\,dx = \sum_{n=-\infty}^{\infty} 4\,|\sin nh|^2 \left|\widehat{f}(n)\right|^2,$$

where $g_h(x) := f(x+h) - f(x-h)$, we have

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2 \left|\widehat{f}(n)\right|^2 \ll \int_0^{2\pi} |f(x+h) - f(x-h)|^2\,dx \ll_\alpha |h|^{2\alpha} \quad \text{(by the Hölder condition).}$$

Now given $p \in \mathbb{N}$, if we choose $h = \pi/2^{p+1}$, then for any $n \in \mathbb{Z}$ s.t. $2^{p-1} < |n| \le 2^p$, we have

$$\frac{1}{\sqrt{2}} \le |\sin nh| \le 1,$$

whence

$$\sum_{\substack{n\in\mathbb{Z}\\ 2^{p-1}<|n|\le 2^p}} \left|\widehat{f}(n)\right|^2 \ll \sum_{\substack{n\in\mathbb{Z}\\ 2^{p-1}<|n|\le 2^p}} |\sin nh|^2 \left|\widehat{f}(n)\right|^2 \le \sum_{n=-\infty}^{\infty} |\sin nh|^2 \left|\widehat{f}(n)\right|^2 \ll_\alpha |h|^{2\alpha} \ll_\alpha \frac{1}{2^{2\alpha p}}.$$

Hence,

$$\sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| = \left|\widehat{f}(0)\right| + \sum_{p=1}^{\infty} \sum_{\substack{n\in\mathbb{Z}\\ 2^{p-1}<|n|\le 2^p}} \left|\widehat{f}(n)\right| \le \left|\widehat{f}(0)\right| + \sum_{p=1}^{\infty} \sqrt{\left(\sum_{\substack{n\in\mathbb{Z}\\ 2^{p-1}<|n|\le 2^p}} 1^2\right)\left(\sum_{\substack{n\in\mathbb{Z}\\ 2^{p-1}<|n|\le 2^p}} \left|\widehat{f}(n)\right|^2\right)}$$

$$\ll_\alpha \left|\widehat{f}(0)\right| + \sum_{p=1}^{\infty} \sqrt{2^p \cdot \frac{1}{2^{2\alpha p}}} = \left|\widehat{f}(0)\right| + \sum_{p=1}^{\infty} \frac{1}{2^{(\alpha-0.5)p}} \ll_{\alpha,f} 1 \quad \text{whenver } \alpha > 0.5.$$

The result follows.

Ch4, Ex1. (a). (2 marks) Let $\gamma : [a,b] \to \mathbb{R}^2$ be a parametrization for Γ. By the convention of the textbook, this implies $\gamma \in \mathcal{C}^1$ and $\gamma'(t) \neq (0,0)$ for all $t \in [a,b]$.
($\Rightarrow$) If γ is a parametrization by arc-length, then by definition $|\gamma'| \equiv 1$ on $[a,b]$, so

$$\int_a^s |\gamma'(t)|\,dt = \int_a^s 1dt = s - a.$$

($\Leftarrow$) Suppose for all $s \in [a,b]$ we have

$$\int_a^s |\gamma'(t)|\,dt = s - a.$$

Since $\gamma \in \mathcal{C}^1$, we see that $|\gamma'| : [a,b] \to \mathbb{R}$ is a continuous function. Therefore, by the fundamental theorem of calculus, for all $s \in [a,b]$ we have

$$|\gamma'(s)| = \frac{d}{ds}\int_a^s |\gamma'(t)|\,dt = \frac{d}{ds}(s-a) = 1.$$

(b). (3 marks) Let $\eta : [a,b] \to \mathbb{R}^2$ be a parametrization for Γ. Again we require $\eta \in \mathcal{C}^1$ and $\eta'(t) \neq (0,0)$ for all $t \in [a,b]$. Therefore, by applying the maximum-minimum theorem to $|\eta'|$, we see that there exists a $C > 0$ s.t. $|\eta'| > C$ on $[a,b]$. This shows that the function

$$h(s) := \int_a^s |\eta'(t)|\, dt$$

on $[a,b]$ is strictly monotone. As a result, by a theorem in math2060, we have

$$\frac{d}{d\xi} h^{-1}(\xi) = \frac{1}{h'(h^{-1}(\xi))} = \frac{1}{|\eta'(h^{-1}(\xi))|}.$$

Therefore, for $\gamma : [0, h(b)] \to \mathbb{R}^2$ defined by $\gamma(\xi) := \eta(h^{-1}(\xi))$, we have

$$\gamma([0, h(b)]) = \eta([a,b]) = \Gamma, \qquad |\gamma'(\xi)| = \left| \eta'(h^{-1}(\xi)) \cdot \frac{d}{d\xi} h^{-1}(\xi) \right| = 1,$$

and $\gamma \in \mathcal{C}^1$.[§] The result follows.

[§]Since h^{-1} is differentiable, it is continuous. Therefore $\gamma' = \dfrac{\eta' \circ h^{-1}}{|\eta' \circ h^{-1}|}$ is a continuous function.