TA's solution* to 3093 assignment 10

Ch5, Ex15. (5 marks)

(a) Applying the Poisson summation formula to $\widehat{g}$, and noting that $\widehat{\widehat{g}}(x) = g(-x)$, we have

$$\sum_{n=-\infty}^{\infty} \left(\frac{\sin \pi(\alpha+n)}{\pi(\alpha+n)}\right)^2 = \sum_{n=-\infty}^{\infty} \widehat{g}(\alpha+n) = \sum_{n=-\infty}^{\infty} g(-n)e^{2\pi i n\alpha} = 1.$$

The result follows.

Alternative

15(a). (p.165) Let $f(x) = g(x)e^{-2\pi i x\alpha}$, where g is the function in Exercise 2. Then we have

$$\widehat{f}(\xi) = \widehat{g}(\xi+\alpha) = \left(\frac{\sin \pi(\xi+\alpha)}{\pi(\xi+\alpha)}\right)^2.$$

By the Poisson summation Formula,

$$1 = \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) = \sum_{n=-\infty}^{\infty} \left(\frac{\sin(n\pi+\alpha\pi)}{\pi(n+\alpha)}\right)^2.$$

Hence,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

(b)

15(b). Since the equality is 1-periodic, it's sufficient to prove for the case $0 < \alpha < 1$. If $\alpha \neq \frac{1}{2}$, we have

$$\int_{\frac{1}{2}}^{\alpha} \frac{\pi^2}{(\sin \pi x)^2} dx = [(-\pi)\cot \pi x]_{\frac{1}{2}}^{\alpha} = -\frac{\pi}{\tan \pi\alpha}.$$

If we let $h_k(x) = \sum_{n=-k}^{k} \frac{1}{(n+x)^2}$,then $|h_k(x)| \leq \frac{\pi^2}{(\sin \pi x)^2}$ which is an integrable function. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{k\to\infty} \int_{\frac{1}{2}}^{\alpha} h_k(x)dx = \int_{\frac{1}{2}}^{\alpha} \frac{\pi^2}{(\sin \pi x)^2} dx = -\frac{\pi}{\tan \pi\alpha}.$$

On the other hand,

$$\lim_{k\to\infty} \int_{\frac{1}{2}}^{\alpha} h_k(x)dx = \sum_{n=-\infty}^{\infty} \int_{\frac{1}{2}}^{\alpha} \frac{1}{(n+x)^2} dx = -\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha}.$$

So we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan \pi\alpha}.$$

If $\alpha = \frac{1}{2}$, we can see easily that $\sum_{-N}^{N} \frac{1}{n+\frac{1}{2}} = \frac{1}{N+\frac{1}{2}}$. Hence, $\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = 0 = \lim_{\alpha\to 1/2} \frac{\pi}{\tan \pi\alpha}$. The formula continues to hold.

*This solution is adapted from the work by former TAs.

We make two remarks. Firstly, the function $f(x) = \frac{\pi^2}{(\sin \pi x)^2}$ is not integrable on $[0,1]$, because when $x \approx 0$, we have

$$\frac{1}{(\sin \pi x)^2} \approx \frac{1}{(\pi x)^2}$$

and

$$\int_0^1 \frac{1}{x^2} = \infty.$$

Secondly, besides the dominated convergence theorem, we can also use the monotone convergence theorem. Alternatively, given a fixed $\alpha \in (0,1)$, by Weierstrass M-Test, $\sum \frac{1}{(n+x)^2}$ is uniformly convergent for x between α and $1/2$, whence justifying $\int \sum = \sum \int$.

Ex19. (5 marks)

(a) The result follows plainly from the Poisson summation formula $\sum f(n) = \sum \widehat{f}(n)$.

(b) By the geometric series formula, for $t \in (0,1)$ we have

$$\frac{t}{t^2+n^2} = \frac{t}{n^2}\frac{1}{1+\left(\frac{t}{n}\right)^2} = \frac{t}{n^2}\sum_{\ell=0}^{\infty}\left(-\frac{t^2}{n^2}\right)^{\ell} = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{n^{2m}}t^{2m-1}.$$

Noting that for $t \in (0,1)$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|(-1)^{m+1}\frac{1}{n^{2m}}t^{2m-1}\right| = \frac{1}{t}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{t^2}{n^2}\right)^m = \frac{1}{t}\sum_{n=1}^{\infty}\frac{t^2/n^2}{1-t^2/n^2}$$
$$= t\sum_{n=1}^{\infty}\frac{1}{n^2-t^2} < \infty,$$

whence we can interchange the order of summation (a proof of this result is given at the end). Consequently,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{n^{2m}}t^{2m-1} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+1}\frac{1}{n^{2m}}t^{2m-1} = \sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1},$$

whence

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}.$$

On the other hand,

$$\sum_{n=-\infty}^{\infty}e^{-2\pi t|n|} = 2\sum_{n=0}^{\infty}e^{-2\pi tn} - 1 = \frac{2}{1-e^{-2\pi t}} - 1.$$

(c) We have

$$\frac{2}{1-e^{-2\pi t}} - 1 = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}$$
$$\Rightarrow \frac{-2\pi t}{e^{-2\pi t}-1} - \pi t = 1 + 2\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m}$$
$$\Rightarrow 1 + \pi t + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}(-2\pi t)^{2m} - \pi t = 1 + 2\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m}.$$

Comparing the coefficients of t on both sides, we get

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

About the change of order of summation in part (b)

Suppose $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|a_{n,m}| := A < \infty$. Here we want to show

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{n,m}. \tag{1}$$

Let $\sigma_n := \sum_{m=1}^{\infty}a_{n,m}$ and $S_n := \sum_{m=1}^{\infty}|a_{n,m}|$. Now $\sum_n S_n = A < \infty$, implying $S_n < \infty$ for each n. As absolute convergence implies convergence, we see that each σ_n is the limit of a convergent series. Also, $\sum_n \sigma_n$ is absolutely convergent because $\sum_n |\sigma_n| \le \sum_n S_n < \infty$. Therefore $\sum_n \sigma_n$ is convergent, showing that the L.H.S. of (1) is meaningful.

On the other hand, since $\sum_{m=1}^{M}\sum_{n=1}^{\infty}|a_{n,m}| = \sum_{n=1}^{\infty}\sum_{m=1}^{M}|a_{n,m}| \le A$, letting $M \uparrow \infty$ we also have $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{n,m}| \le A < \infty$. Therefore by symmetry the R.H.S. of (1) is meaningful too.

Fix an $\varepsilon > 0$. By $\sum_n S_n < \infty$, there exists N s.t.

$$\sum_{n=N+1}^{\infty} S_n < \varepsilon.$$

For this N, since $S_n < \infty$ for all $1 \le n \le N$, there exists M s.t. for all $1 \le n \le N$, we have

$$\sum_{m=M+1}^{\infty}|a_{n,m}| < \frac{\varepsilon}{N}.$$

Consequently,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{n,m} - \sum_{n=1}^{N}\sum_{m=1}^{M}a_{n,m}\right| &\le \left|\sum_{n=1}^{\infty}\sigma_n - \sum_{n=1}^{N}\sigma_n\right| + \left|\sum_{n=1}^{N}\sigma_n - \sum_{n=1}^{N}\sum_{m=1}^{M}a_{n,m}\right| \\
&= \left|\sum_{n=N+1}^{\infty}\sigma_n\right| + \left|\sum_{n=1}^{N}\left(\sum_{m=1}^{\infty}a_{n,m} - \sum_{m=1}^{M}a_{n,m}\right)\right| \\
&\le \sum_{n=N+1}^{\infty}S_n + \sum_{n=1}^{N}\left|\sum_{m=M+1}^{\infty}a_{n,m}\right| \le 2\varepsilon.
\end{aligned}$$

By using the same argument, there exist some $N' > N$ and $M' > M$ such that

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{n,m} - \sum_{m=1}^{M'}\sum_{n=1}^{N'}a_{n,m}\right| \le \varepsilon.$$

Since we can interchange the order of summation for finite sums, we have

$$\begin{aligned}
\left|\sum_{m=1}^{M'}\sum_{n=1}^{N'} a_{n,m} - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m}\right| &= \left|\sum_{n=1}^{N'}\sum_{m=1}^{M'} a_{n,m} - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m}\right| \\
&\leq \left|\sum_{n=N+1}^{N'}\sum_{m=1}^{M'} a_{n,m}\right| + \left|\sum_{n=1}^{N}\sum_{m=1}^{M'} a_{n,m} - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m}\right| \\
&\leq \sum_{n=N+1}^{N'}\sum_{m=1}^{M'} |a_{n,m}| + \sum_{n=1}^{N}\sum_{m=M+1}^{M'} |a_{n,m}| \\
&\leq \sum_{n=N+1}^{\infty} S_n + \sum_{n=1}^{N} \frac{\varepsilon}{N} \leq 2\varepsilon.
\end{aligned}$$

This gives

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}\right| \\
&\leq \left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m}\right| + \left|\sum_{n=1}^{N}\sum_{m=1}^{M} a_{n,m} - \sum_{m=1}^{M'}\sum_{n=1}^{N'} a_{n,m}\right| + \left|\sum_{m=1}^{M'}\sum_{n=1}^{N'} a_{n,m} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{n,m}\right| \\
&\leq 5\varepsilon.
\end{aligned}$$

The result follows.