THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH1010 University Mathematics 2022-2023 Term 1

Suggested Solutions for Quiz 2

1. (25 marks) Define $f(x) = xe^{-\frac{x}{5}}$, with domain $D_f = [1, \infty)$.

   (a) Find all the local maximum and local minimum (if exist) of $f(x)$. Show steps.

   (b) Find all the point(s) of inflection (if exist) of $f(x)$. Show steps.

   (c) Find the global maximum and minimum values (if exist) of $f(x)$. Explain briefly.

Solution

(a) First we calculate the first derivative of f.

$$f'(x) = e^{-\frac{x}{5}} + xe^{-\frac{x}{5}}\left(-\frac{1}{5}\right)$$
$$= e^{-\frac{x}{5}}\left(1 - \frac{x}{5}\right).$$

Thus $f' < 0$ on $(1, 5)$, and $f' > 0$ on $(5, \infty)$. This means f is initially increasing on $(1, 5)$, then decreasing on $(5, \infty)$.

Hence $x = 5$ is the only local maximum (with maximum value $f(5) = \frac{5}{e}$), and $x = 1$ (edge point) is the only local minimum (with minimum value $f(1) = e^{-\frac{1}{5}}$).

(b)

$$\begin{aligned} f''(x) &= -\frac{1}{5}e^{-\frac{x}{5}}\left(1-\frac{x}{5}\right)-\frac{1}{5}e^{\frac{x}{5}} \\ &= e^{-\frac{x}{5}}\left(-\frac{1}{5}\right)\left(1-\frac{x}{5}+1\right) \\ &= -\frac{1}{5}e^{-\frac{x}{5}}\left(2-\frac{1}{5}x\right) \qquad (1) \end{aligned}$$

Set $f''(x) = 0$, we get $x = 10$(since $e^{-\frac{x}{5}} > 0$).

x	[1,10)	10	$[10, \infty)$
$f''(x)$	-ve	0	+ve

So the only point of inflection is attained at $x = 10$.

The point of inflection is $(10, \frac{10}{e^2})$.

(c) From (a), we know that f attains its global maximum at $x = 5$ implies Global maximum value of $f(x) = \frac{5}{e}$.

Consider $f(1) = \frac{1}{e^{1/5}} > 0$, and f is strictly decreasing on $(5, \infty)$, we just need to check the following limit for comparison:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^{x/5}} = 0$$

but $\frac{x}{e^{x/5}} \neq 0 \quad \forall x \in [1, \infty)$.

Hence, f has no global minimum.

2. (25 marks) Evaluate each of the following limits by using L'Hôpital's Rule. **NO** marks will be given if you use other methods or approaches. Show all steps clearly.

   (a) $\lim_{x \to 0} \dfrac{\cos x - 1}{x^2}$

   (b) $\lim_{x \to 0^+} \dfrac{\sqrt{x}}{1 - e^{2\sqrt{x}}}$

   (c) $\lim_{x \to 0} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x} \right)$

   (d) $\lim_{x \to 0^+} x \cdot (\ln x)^2$

   (e) $\lim_{x \to \infty} (e^{3x} - 2x)^{\frac{1}{x}}$

Solution

(a)

$$\begin{aligned} &\lim_{x \to 0} \frac{\cos x - 1}{x^2} \qquad \left(\frac{0}{0}\right) \\ =&\lim_{x \to 0} \frac{-\sin x}{2x} \\ =&\lim_{x \to 0} -\frac{1}{2}\frac{\sin x}{x} \\ =&-\frac{1}{2} \end{aligned}$$

(b) The form is $\frac{0}{0}$, thus L'Hôpital's rule applies

$$\begin{aligned} &\lim_{x \to 0^+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}} \\ =&\lim_{x \to 0^+} \frac{\frac{1}{2\sqrt{x}}}{0 - e^{2\sqrt{x}} \cdot 2 \cdot \frac{1}{2\sqrt{x}}} \\ =&\lim_{x \to 0^+} \frac{1}{-2e^{2\sqrt{x}}} \\ =&-\frac{1}{2} \end{aligned}$$

(c) The form is $\infty - \infty$, however, if the difference is written as a single fraction, then

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-e^x+1}{xe^x-x}.$$

This gives us the indeterminate form $\frac{0}{0}$. Thus, L'Hôpital's rule applies

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)&=\lim_{x\to 0}\frac{x-e^x+1}{xe^x-x}\\&=\lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1}\\&=\lim_{x\to 0}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}.\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 0^+} x\cdot(\ln x)^2&=\lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}} && \left(\frac{\infty}{\infty}\right)\\&=\lim_{x\to 0^+}\frac{2(\ln x)\frac{1}{x}}{-\frac{1}{x^2}}\\&=\lim_{x\to 0^+}-2x\ln x\\&=\lim_{x\to 0^+}\frac{-2\ln x}{\frac{1}{x}} && \left(\frac{\infty}{\infty}\right)\\&=\lim_{x\to 0^+}\frac{-\frac{2}{x}}{-\frac{1}{x^2}}\\&=\lim_{x\to 0^+}2x=0.\end{aligned}$$

(e) Since $\lim_{x\to\infty}(e^{3x}-2x)=\infty$ and $\lim_{x\to\infty}\frac{1}{x}=0$, the limit takes the indeterminate form ∞^0.

Let $f(x)=(e^{3x}-2x)^{\frac{1}{x}}$, then

$$\ln f(x)=\frac{\ln(e^{3x}-2x)}{x}.$$

which assumes the indeterminate form $\frac{\infty}{\infty}$ and L'Hôpital's rule applies,

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln(e^{3x} - 2x)}{x} = \lim_{x\to\infty} \frac{3e^{3x} - 2}{e^{3x} - 2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{9e^{3x}}{3e^{3x} - 2} = \lim_{x\to\infty} \frac{27e^{3x}}{9e^{3x}} = 3.$$

By continuity of exponential function,

$$\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} \ln f(x)} = e^3.$$

3. (20 marks)

(a) Let $y = e^x \cos(\sin x)$, find $y(0),\ y'(0)$ and $y''(0)$ respectively.

(b) Find the n-th derivative of $y = x^2 a^x$, where $a > 1$ is a constant.

Solution

(a) Differentiation gives

$$y = e^x \cos(\sin x),$$

$$\begin{aligned} y' &= e^x \cos(\sin x) - e^x \sin(\sin x)\ \cos x \\ &= y - y \tan(\sin x)\ \cos x \\ &= y\,(1 - \tan(\sin x)\ \cos x) \end{aligned}$$

and

$$\begin{aligned} y'' = &\ y'(1 - \tan(\sin x)\ \cos x) \\ &- y\left(\sec^2(\sin x)\ \cos^2 x - \tan(\sin x) \sin x\right). \end{aligned}$$

Hence,

$$y(0) = 1,\ y'(0) = 1,\ y''(0) = 0.$$

(b) Since $\dfrac{d^r}{dx^r} a^x = a^x (\ln a)^r$ for $r = 0, 1, 2, \cdots, n,$ therefore,

$$\begin{aligned} &\frac{d^n y}{dx^n} \\ &= x^2 a^x (\ln a)^n + C_1^n 2x) a^x (\ln a)^{n-1} + C_2^n (2) a^x (\ln a)^{n-2} \\ &= a^x (\ln a)^{n-2} \left(x^2 (\ln a)^2 + 2nx(\ln a) + n(n-1)\right) \end{aligned}$$

4. (20 marks)

(a) Find the 5th order Taylor polynomial for $f(x) = \arctan x$ about $x = 0$. Show all steps.

(b) Using (a), find, with steps, the first 3 non-zero terms of the Taylor polynomial (in ascending order) for $g(x) = \dfrac{1117x^{22}}{2+2x^2}$ about $x = 0$.

(**Note: NO** marks will be given if you simply write down the final answer for (a), and if you do not use the result of (a) when attempting (b).)

Solution

(a) The 5th order Taylor polynomial for $f(x) = \arctan x$ about $x = 0$ is as follows:

$$
\begin{aligned}
f(x) = \arctan x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 \\
+ \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 \\
+ \frac{f^{(5)}(0)}{5!}(x-0)^5
\end{aligned}
$$

i. $f(0) = 0$

ii. $f'(x) = \dfrac{1}{1+x^2} \implies f'(0) = \dfrac{1}{1+0^2} = 1$

iii. $f''(x) = -1(1+x^2)^{-2} \cdot 2x = -2x(1+x^2)^{-2} \implies f''(0) = 0$

iv. $f'''(x) = -2(1+x^2)^{-2} + 4x(1+x^2)^{-3} \cdot 2x \implies f'''(0) = -2$

v.

$$\begin{aligned} f^{(4)}(x) &= 4(1+x^2)^{-3}\cdot 2x + 4(1+x^2)^{-3}\cdot 2\cdot 2x - 24x^2(1+x^2)^{-4}\cdot 2x \\ &= 8x\cdot(1+x^2)^{-3} + 16x\cdot(1+x^2)^{-3} - 48x^3\cdot(1+x^2)^{-4} \\ &= 24x(1+x^2)^{-3} - 48x^3(1+x^2)^{-4} \implies f^{(4)}(0) = 0 \end{aligned}$$

vi.

$$\begin{aligned} f^{(5)}(x) &= 24(1+x^2)^{-3} - 72x\cdot(1+x^2)^{-4}\cdot(2x) - 144x^2\cdot(1+x^2)^{-4} \\ &\quad + 192x^3(1+x^2)^{-5}\cdot(2x) \\ &\implies f^{(5)}(0) = 24 \end{aligned}$$

Therefore, the 5th order Taylor polynomial for $f(x) = \arctan x$ about $x = 0$ is

$$\begin{aligned} &= x - \frac{2}{3!}x^3 + \frac{24}{5!}x^5 \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \end{aligned}$$

(b) By using (a), we have

$$f(x) = \arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \mathrm{O}\left(x^7\right),$$

Consider

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 + \mathrm{O}\left(x^6\right)$$

Note that

$$\begin{aligned}
g(x) &= \frac{1117x^{22}}{2+2x^2} \\
&= \frac{1117x^{22}}{2}\left(\frac{1}{1+x^2}\right) \\
&= \frac{1117x^{22}}{2}\left(1 - x^2 + x^4 + \mathrm{O}\left(x^6\right)\right) \\
&= \frac{1117x^{22}}{2} - \frac{1117x^{24}}{2} + \frac{1117x^{26}}{2} + \mathrm{O}\left(x^{28}\right)
\end{aligned}$$

Thus, the first 3 non-zero terms of the Taylor polynomial for $g(x)$ are $\frac{1117x^{22}}{2}$, $-\frac{1117x^{24}}{2}$ and $\frac{1117x^{26}}{2}$.

5. (10 marks) Show that

$$\frac{2(b-a)}{1+4b^2} < \arctan(2b) - \arctan(2a) < \frac{2(b-a)}{1+4a^2} \qquad \text{for } 0 \le a < b.$$

State the theorem(s) that you have used.

Solution

Let $f(x) = \arctan(2x)$, which is differentiable for $x > 0$. By the mean value theorem, there exists $c \in (a, b)$ such that

$$\frac{\arctan(2b) - \arctan(2a)}{b-a} = f'(c),$$

that is,

$$\frac{\arctan(2b) - \arctan(2a)}{b-a} = \frac{2}{1+4c^2}.$$

For $a < c < b$,

$$\frac{2}{1+4b^2} < \frac{2}{1+4c^2} < \frac{2}{1+4a^2},$$

thus

$$\frac{2(b-a)}{1+4b^2} < \arctan(2b) - \arctan(2a) < \frac{2(b-a)}{1+4a^2},$$

for $0 \le a < b$.