MATH5070
Homework 4 solution

1. (i) ($\Longrightarrow$) Suppose $\pi : E \to M$ is a trivial bundle with global trivialization $\Phi : E \to M \times \mathbb{R}^k$. Let $\{e_1, \dots, e_k\}$ be the standard basis of $\mathbb{R}^k$. We define $s_i : M \to E$ for each $i = 1, \dots, k$ as:
$$s_i(p) = \Phi^{-1}(p, e_i).$$
It is easy to check s_i is a smooth section for each i. Since $\{e_1, \dots, e_k\}$ is a basis of $\mathbb{R}^k$ and $\Phi\mid_{E_p}: E_p \to \mathbb{R}^k$ is linear isomorphism, $s_1(p), \dots, s_k(p)$ form a basis of E_p.
($\Longleftarrow$) Suppose $\pi : E \to M$ admits global frame $s_1, \dots, s_k$. For any $x \in E$, let $p = \pi(x)$, then $x \in E_p$. Since $\{s_1(p), \dots, s_k(p)\}$ is a basis of E_p, x can be uniquely represented as $x = a_1 s_1(p) + \cdots + a_k s_k(p)$. We define $\Phi : E \to M \times \mathbb{R}^k$ by
$$\Phi(x) = (p, a_1, \dots, a_k).$$
Clearly Φ is a smoth diffeomorphism and Φ_{E_p} is a linear isomorphism.

   (ii) Let $\{v_1, \dots, v_k\}$ be a basis of T_eG. We claim that the left-invariant vector field $V_1, \dots, V_k$ is a global frame of TG, where $V_i(g) = dL_g|_e(v_i)$ for any $g \in G$. It suffices to show that $V_1(g), \dots, V_k(g)$ form a basis of T_gG. In fact, $Lg : G \to G$ is a diffeomorphism, then dL_g is a linear isomorphism. Thus $V_1(g), \dots, V_k(g)$ form a basis of T_gG. By Q2(i), TG is a trivial bundle.

2. Let $\{U_\alpha, \alpha \in I\}$ be an open covering of M, and for each $\alpha \in I$ let $\{s_1^\alpha, \dots, s_k^\alpha\}$ ba a local frame. We define the inner product $\langle , \rangle_\alpha$ on E_{U_α} by $\langle s_i^\alpha(p), s_j^\alpha(p)\rangle = \delta_{ij}$ for any $p \in U_\alpha$. We can check that $\langle , \rangle_\alpha$ is smooth on E_α. Let $\{\rho_\alpha, \alpha \in I\}$ be a partition of unity subordinate to the above covering. Consider the sum $\sum \rho_\alpha \langle , \rangle_\alpha$. Firstly, it is well-defined by the local finiteness of partition of unity. Secondly, it is smooth since $\langle , \rangle_\alpha$ is smooth for any $\alpha \in I$. Finally, it suffices to show that $\sum \rho_\alpha \langle , \rangle_\alpha$ is a inner product. Symmetry and bilinearity follow directly from that of $\langle , \rangle_\alpha$. Positive-definiteness follows from that of $\langle , \rangle_\alpha$ plus the fact that $\rho_\alpha \geq 0$ for all α and there exists at least one $\rho_\alpha(p) > 0$ for any $p \in M$.

3. Let $\pi' : E^* \to M$ be the dual bundle of $\pi : E \to M$, say it is of rank k. Since M is compact, we can choose the local trivializations $\Phi_i : \pi'^{-1}(U_i) \to U_i \times \mathbb{R}^k$ for $i = 1, \dots, r$ such that $\{U_i\}_{i=1}^r$ covers M. Let $\{\sigma_1^i, \dots, \sigma_k^i\}$ be a local frame over each U_i. Let $\{\rho_i\}_{i=1}^r$ be a partition of unity subordinate to the above covering. We extend the section σ_j^i to a global section s_j^i as follows:
$$s_j^i = \begin{cases} \rho_i(p)\sigma_j^i(p) & p \in U_i, \\ 0 \quad p \in M \setminus \text{supp } \rho_i. \end{cases}$$
We rewrite these global sections as $s_1, \dots, s_N$, and it can be checked that for every $p \in M$, $s_1(p), \dots, s_N(p)$ span E_p^* by the property of partition of unity.
We define the map $\Psi : E \to M \times \mathbb{R}^N$ by:
$$\Psi(v) = (\pi(v), s_1(\pi(v))v, \dots, s_N(\pi(v))v).$$
We see that $\Psi|_{E_p}$ is linear since $s_j(p)$ is linear for all $j = 1, \dots, N$, and $\Psi|_{E_p}$ is injective since $s_1(p), \dots, s_N(p)$ span E_p^* for all p. Thus we can identify E with its image which is a subbundle of $M \times \mathbb{R}^N$.

4. By Q3, $E \to M$ is a subbundle of the trivial bundle $M \times \mathbb{R}^N \to M$. Choose an inner product on the trivial bundle. There is an orthogonal complement F_p of E_p for all $p \in M$. Take $F = \cup_{p \in M} F_p$, then $F \to M$ is a vector bundle such that $E \oplus F$ is the trivial bundle $M \times \mathbb{R}^N \to M$. To show that $F \to M$ is a vector bundle, we need to check the local trivializations. For any point $p \in M$, there exists an open neighborhood U of p which admits a local frame $\{s_1, \ldots, s_k\}$ for E. Then we can extend $\{s_1(p), \ldots, s_k(p)\}$ to a basis of $\mathbb{R}^N$, say $\{s_1(p), \ldots, s_k(p), s_{k+1}(p), \ldots, s_N(p)\}$. We define local section $s_j : U \to \mathbb{R}^N$ to be constant (i.e., $s_j(q) = s_j(p)$ for any $q \in U$) for $j = k+1, \ldots, N$. We can choose U to be small enough such that $\{s_1, \ldots, s_k, s_{k+1}, \ldots, s_N\}$ forms a local frame, and we use Gram-Schmidt's method to orthogonalize $\{s_1, \ldots, s_k, s_{k+1}, \ldots, s_N\}$. Then it gives us a local trivialization of $F \to M$. Also we can check the transition maps smoothly depends on $p \in M$.

5. f^*E can be identified as a subspace of $M \times E$, that is $f^*E = \{(p, v) \in M \times E \mid \pi(v) = f(p)\}$. The projection map $\pi' : f^*E \to M$ is just the projection to the first component. Firstly, we need to find the local trivalization. For any point $p \in M$, there exists an open neighborhood U of $f(p)$ such that there exists a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. $V = f^{-1}(U)$ is an open neiborhood of p. We define the map $\Psi : f^*E|_V \to V \times \mathbb{R}^k$ by
$$\Psi(p, v) = (p, \mathrm{proj}_2(\Phi(v))),$$
where $\mathrm{proj}_2 : U \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection to the second component. Clearly, Ψ and its inverse Ψ^{-1} are smooth, where $\Psi^{-1}(p, w) = (p, \Phi^{-1}(f(p), w))$. Then we need to show that the tansition maps smoothly depend on $p \in U_1 \cap U_2$. Let $g : U_1 \cap U_2 \to GL(\mathbb{R}^k)$ is a transition function for N. Then $g \circ f : f^{-1}(U_1) \cap f^{-1}(U_2) \to GL(\mathbb{R}^k)$ is the transition function for f^*E, which is linear and smoothly depends on $f^{-1}(p) \in f^{-1}(U_1) \cap f^{-1}(U_2)$.

6. We use the notations in HW1. Firstly, we need to find the local trivializations. E can be written as $E = \{(V, v) \in M_k(\mathbb{R}^n) \times \mathbb{R}^n \mid v \in V\}$. $\{\mathcal{U}_I\}$ is an open covering of $M_k(\mathbb{R}^n)$ as in HW1. For any $\mathcal{U}_I$,
$$E_{\mathcal{U}_I} = \{(V, v) \in \mathcal{U}_I \times \mathbb{R}^n \mid v \in V\} = \{(A^\#, A^\# x) \mid A \in \mathcal{M}_I, x \in \mathbb{R}^k\}.$$
We define $\Psi : \mathcal{U}_I \times \mathbb{R}^k \to E_{\mathcal{U}_I}$ by
$$\Psi(A^\#, x) = (A^\#, A^\# x).$$
It can be checked that Ψ is bijective. Therefore, Ψ^{-1} gives a local trivialization over $\mathcal{U}_I$. It can also be checked that transition maps are linear and soomthly depend on $V \in M_k(\mathbb{R}^n)$.

7. By Q3, there exists an integer N such that $E \to M$ is a subbundle of the trivial bundle $M \times \mathbb{R}^N \to M$. For each $p \in M$, E_p is a k-dimentional vector subspace of $\mathbb{R}^N$. We define $f : M \to M_k(\mathbb{R}^N)$ by $f(p) = E_p$, then $E = f^*E_{can}$.