Homework 2 for MATH5070
Topology of Manifolds
Due Wednesday, Oct. 5

1. Let $\mathcal{M}_n$ be the space of all $n \times n$ real matrices and Sym_n be the space of all $n \times n$ symmetric matrics. Consider the map

$$f : \mathcal{M}_n \to \mathrm{Sym}_n,\ A \mapsto f(A) = A^t A.$$

(i) Since both $\mathcal{M}_n$ and Sym_n are linear spaces, we can identify $T_A\mathcal{M}_n$ with $\mathcal{M}_n$ and $T_{f(A)}\mathrm{Sym}_n$ with Sym_n. Show that $df_A(B) = A^t B + B^t A$.

(ii) Prove that $I_n \in \mathrm{Sym}_n$ is a regular value of f.

(iii) Conclude that $O(n)$ is a $\frac{n(n-1)}{2}$ dimensional submanifold of $\mathcal{M}_n$.

(iv) Find all regular points, critical points, regular values and critical values of f.

(v) Check Sard's theorem for this example.

2. The Whitney embedding theorem says that if M is an n-dimensional manifold, then there exists an embedding $\iota : M \to \mathbb{R}^{2n}$, i.e., every n-dimensional manifold is diffeomorphic to a submanifold of Euclidean $2n$ dimensional space. We prove an easier version of the theorem.

Theorem Let M be a compact manifold. Then M can be embedded in some Euclidean space.

Hint: Let $\mathcal{A} = \{(\varphi_i, U_i, V_i),\ i = 1, \cdots, r\}$ be an atlas. Let $\{\rho_i,\ i = 1, \cdots, r\}$ be a partition of unity subordinate to this atlas. For each i, let $\psi_i : M \to \mathbb{R}^n$ be the map

$$\psi_i(p) = \begin{cases} \rho_i(p)\varphi_i(p) & \text{if } p \text{ is in } U_i \\ 0 & \text{if } p \text{ is not in } U_i. \end{cases}$$

Show that the map

$$\iota : M \to \mathbb{R}^{nr+r}$$

which maps $p \in M$ to the $(nr + r)$-tuple

$$(\psi_1(p), \cdots, \psi_r(p), \rho_1(p), \cdots \rho_r(p))$$

is an embedding.

3.(optional) Suppose that v and w are two non-vanishing smooth vector fields which are pointwise proportional, that is, $w = f \cdot v$ for a non-vanishing smooth function f on M. Prove that the respective maximal integral curves $\gamma : I \to M$ and $\tilde{\gamma} : J \to M$ for v and w through $p \in M$ at time 0 satisfy $\tilde{\gamma} = \gamma \circ F$ for a unique diffeomorphism $F : J \to I$ preserving 0.

Roughly speaking, this statement says that maximal integral curves are "the same" up to a reparametrization. In other words, the trajectory of the integral curve depends only on the direction of the vector field.

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