

Homework 1 for MATH5070

Topology of Manifolds

Due Wednesday, Sept. 21

The goal of the exercises below is to show that the set of all k -dimensional subspace of \mathbb{R}^n is a manifold of dimension $k(n - k)$.

- (i) For $0 < k < n$ let M be the set of all injective linear mappings of \mathbb{R}^k into \mathbb{R}^n . Show that M is a kn -dimensional manifold. *Hint:* It's an open subset of a kn -dimensional vector space.
(ii) Define E to be the set of all pairs, $(L_1, L_2) \in M \times M$ with $\text{Image}L_1 = \text{Image}L_2$. Show that E is an equivalence relation. What are the equivalence classes?

Let M_E be the set of equivalence classes. I want you to show that M_E is a $k(n - k)$ -dimensional manifold. Here are some hints:

- Let \mathcal{M} be the set of real $n \times k$ matrices of rank k . Identify \mathcal{M} with M . If A_1 and A_2 are elements of \mathcal{M} show that the following three conditions are equivalent
 - A_1 is E -equivalent to A_2 .
 - There exists an invertible $k \times k$ matrix, B , such that $A_1 = A_2B$.
 - A_1 can be obtained from A_2 by a sequence of elementary column operations.
- Let $I = (i_1, i_2, \dots, i_k)$ be a k -tuple of integers with $1 \leq i_1 < i_2 < \dots < i_k \leq n$. For $A \in \mathcal{M}$ let A_I be $k \times k$ minor of A whose rows are the i_1 -th, i_2 -th, \dots , i_k -th rows of A . Let

$$\mathcal{M}_I = \{A \in \mathcal{M}, A_I \text{ is invertible}\}.$$

Prove:

- \mathcal{M}_I is an open subset of \mathcal{M} .
 - $\mathcal{M} = \bigcup \mathcal{M}_I$.
 - If A is in \mathcal{M}_I , then every matrix which is E -equivalent to A is in \mathcal{M}_I .
- Prove the following **Lemma**: If A is in \mathcal{M}_I , then there exists a *unique* matrix $A^\#$ in \mathcal{M}_I which is E -equivalent to A and has the property that $A_I^\#$ is the identity matrix.
 - Given $A \in \mathcal{M}$ let $[A]$ be its equivalence class in M_E . Set

$$\mathcal{U}_I = \{[A], A \in \mathcal{M}_I\}$$

and define a bijective map

$$\varphi_I : \mathcal{U}_I \rightarrow \mathbb{R}^{k(n-k)}$$

as follows. Let W_I be the set of $n \times k$ matrices, A , for which A_I is the $k \times k$ identity matrix; and, using the results of problem 4 prove:

Lemma W_I is contained in \mathcal{M}_I and the map

$$W_I \rightarrow \mathcal{U}_I, \quad A \rightarrow [A] \tag{1}$$

is bijective.

Now show that there is a (simple and natural) identification

$$W_I \simeq \mathbb{R}^{k(n-k)} \tag{2}$$

and compose (2) with the inverse of (1).

6. Let \mathcal{A} be the collection of charts $(\varphi_I, \mathcal{U}_I, \mathbb{R}^{k(n-k)})$. Verify that this is an atlas by computing the transition maps associated with any pairs of charts in this collection and verifying that they are smooth.

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