Suggested Solution to midterm exam

1. Let $(X, \mathcal{M}, \lambda)$ be a measure space.

   (a) Let $A_k \in \mathcal{M}, k \geq 1$. Show that the set $E = \{x \in X : x$ belongs to exactly 2022 many $A_k\}$ is measurable.

   (b) Assume that $\sum_k \lambda(A_k) < \infty$. Show that the set $F = \{x \in X : x$ belongs to infinitely many $A_k\}$, is a null set. You may assume F to be measurable.

Solution.

(a) $S : X \to \overline{\mathbb{R}}$ defined by $S(x) := \sum_{k=1}^{\infty} \chi_{A_k}(x)$ is a measurable function, whence $E = S^{-1}(\{2022\})$ is measurable.
Alternatively, since

$$E = \bigcup_{i_1 < \cdots < i_{2022}} \left(A_{i_1} \cap \cdots \cap A_{i_{2022}} \cap \bigcap_{k \notin \{i_1, \ldots, i_{2022}\}} A_k^c \right)$$

and $\left\{(i_1, \ldots, i_{2022}) \in \mathbb{N}^{2022} : i_1 < \cdots < i_{2022}\right\}$ is a subset of the countable set $\mathbb{N}^{2022}$, we see that E is measurable.

(b) (You may also refer to HW1 Q7)
Note that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k.$$

Since $\sum_{k=1}^{\infty} \lambda(A_k) < \infty$, we have $\sum_{k=n}^{\infty} \lambda(A_k) \to 0$ as $n \to \infty$. For any n$\in N$, we have

$$F \subset \bigcup_{k \geq n} A_k$$

and so

$$\lambda(F) \leq \sum_{k=n}^{\infty} \lambda(A_k) \ .$$

Taking $n \to \infty$, we have $\lambda(A) = 0$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Assume that $\mu(X) > 0$. Suppose that f is a measurable function in X which is positive almost everywhere. Show that the set $\{x \in X : f(x) > \rho\}$ has positive measure for some $\rho > 0$.

   (b) Let $f \geq 0$ be integrable and $\int f d\mu = c \in (0, \infty)$. Prove that

$$\lim_{n\to\infty} \int n \log\left(1 + \frac{f}{n}\right) d\mu = c.$$

Solution.

(a) Let E_n=$\{x \in X : f(x) > 1/n\}$ and E=$\{x \in X : f(x) > 0\}$, which are measurable since f is a measurable function.

Note that

$$E = \bigcup_{n=1}^{\infty} E_n = \lim_{n\to\infty} E_n$$

Moreover, E_n is an ascending sequence of measurable sets, we have

$$\mu(E) = \lim_{n\to\infty} \mu(E_n)$$

Because f is positive almost everywhere, $\mu(E) = \mu(X) > 0$

Then there exists k s.t. $\mu(E_k) \geq \frac{1}{2}\mu(X) > 0$

Take $\rho = \frac{1}{k}$, we have $\mu\{x \in X : f(x) > \rho\} > 0$

(b) (You may also refer to HW2 Q6)

Let $g_n(x) = n \log\left(1 + \dfrac{f(x)}{n}\right)$. Since $\displaystyle\int f\, d\mu = c \in (0, \infty)$, we know that $\mu(\{x : f(x) = \infty\}) = 0$ and $\mu(\{x : f(x) > 0\}) > 0$.

Observe that

$$\lim_{n\to\infty} g_n(x) = f(x) a.e.$$

Moreover

$$g_n = n \log\left(1 + \frac{f}{n}\right) \leq n \cdot \frac{f}{n} = f \in L^1(\mu).$$

By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int g_n\, d\mu = \int \lim_{n\to\infty} g_n\, d\mu = \int f\, d\mu = c.$$

3. Let m be a Borel measure and μ_Λ a Riesz measure on $\mathbb{R}^n$.

   (a) Show that μ_Λ satisfies: For each measurable set E and $\varepsilon > 0$, there exist a closed set A and an open set $B, A \subset E \subset B$, such that $\mu_\Lambda(B \backslash A) < \varepsilon$.

   (b) Suppose that m is equal to μ_Λ on all open sets. Show that they coincide on all Borel sets.

   (c) Suppose that m is finite on all compact sets. Show that it is the restriction of some Riesz measure on $\mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$.

Solution. (You may also refer to Lecture Notes Chap 2 and HW5)

(a) It's clear that any Riesz measure μ_Λ on $\mathbb{R}^n$ is σ-finite by Riesz Representation Theorem. Let E be measurable and $E_j = E \cap X_j$ where $\mathbb{R}^n = \bigcup_j X_j$ is a σ-finite decomposition of $\mathbb{R}^n$. By outer regularity of the Riesz measure, for each $\varepsilon > 0$, there exists an open set B_j containing E_j such that $\mu_\Lambda(B_j \backslash E_j) = \mu_\Lambda(B_j) - \mu_\Lambda(E_j) \le \varepsilon/2^j$ for all $j \ge 1$. It follows that $\mu_\Lambda(B \backslash E) \le \sum_j \mu_\Lambda(B_j \backslash E_j) \le \varepsilon$ where $B = \bigcup_j B_j$ is open after using $B \backslash E = \left(\bigcup_j B_j\right) \backslash (\bigcup_k E_k) = \bigcup_j (B_j \backslash \bigcup_k E_k) \subset \bigcup_j (B_j \backslash E_j)$. Next, we apply this result to the complement of E, E^c, to get an open B_0 such that $E^c \subset B_0$ and $\mu(B_0 \backslash E^c) < \varepsilon$. Then the closed set $A = B_0^c$ is contained in E and satisfies $\mu_\Lambda(E \backslash A) = \mu_\Lambda(B_0 \backslash E^c) < \varepsilon$.

(b) Let $E \in \mathcal{B}$. For $\varepsilon > 0$, by (a), there exists an open set A and a closed set B with $A \subset E \subset B$ such that $\mu_\Lambda(B \setminus A) < \varepsilon$. Since B and $B \setminus A$ are open, m and μ_Λ coincide on them, and one has

$$\begin{aligned}\mu_\Lambda(E) = \mu_\Lambda(B) - \mu_\Lambda(B \setminus E) \ge \mu_\Lambda(B) - \mu_\Lambda(B \setminus A) = m(B) - m(B \setminus A)\\ \ge m(E) - \varepsilon.\end{aligned}$$

By changing the position of m and μ_Λ, one has

$$m(E) - \varepsilon \le \mu_\Lambda(E) \le m(E) + \varepsilon.$$

Since this holds for any $\varepsilon > 0$, one has $\mu_\Lambda(E) = m(E)$.

(c) It suffices to show both measures coincide on open sets because then we can apply (b).

For $f \in C_c$, define the linear functional by

$$\Lambda f = \int f dm \ .$$

As m is finite on compact sets, this is a well-defined and obviously a positive functional. By the representation theorem there is a Riesz measure μ_Λ such that

$$\int f dm = \int f \mu_\Lambda \ , \quad \forall f \in C^c(\mathbb{R}^n) \ .$$

For any open set G, we can find an ascending sequence of compact sets $\{K_n\}$ such that $G = \bigcup_n K_n$. Let f_n satisfy $K_n < f_n < G$ so that f_n increases to χ_G pointwisely. By Lebesgue monotone convergence we get

$$m(G) = \lim_{n\to\infty} \int_G f_n dm = \lim_{n\to\infty} \int_G f_n d\mu_\Lambda = \mu_\Lambda(G) \ .$$

4. Let $\mathcal{L}$ denote the Lebesgue outer measure on $\mathbb{R}$.
   (a) Suppose that $B \subset \mathbb{R}$ is not Lebesgue measurable. Show that there exists $\epsilon > 0$ such that for every Lebesgue measurable $A \subset B, \mathcal{L}(B \backslash A) > \epsilon$.
   (b) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Show that $g(E)$ is Lebesgue measurable if $E \subset \mathbb{R}$ is Lebesgue measurable.

Solution.

(a) Prove by contradiction.
Suppose not, than there exist a squence of measurable sets A_n's s.t. $A_n \subset B, \mathcal{L}(B\backslash A_n) \leq \frac{1}{n}$. Let $A = \bigcup_{n=1}^{\infty} A_n$, then $A \subset B$ and $\mathcal{L}(B\backslash A) = 0$. Since the Lebesgue outer measure $\mathcal{L}$ is complete, we have $B\backslash A$ is Lebesgue measurable, which implies $B = (B\backslash A) \bigcup A$ is also measurable. Contradicton arises.

(b) Assume that E is compact first. As the image of a compact set under a continuous map is again compact and so is Borel, we see that $g(E)$ is also compact, hence measurable. Next, let E be a bounded measurable set. By inner regularity we can find a set $F \subset E$ which is the countable union of compact sets satisfying $\mathcal{L}^n(E\backslash F) = 0$. Hence the set $N = E \setminus F$ is null and $g(E) = g(F) \cup g(N)$. We have $g(F) = \bigcup_j g(K_j)$ where K_j are compact, so $g(F)$ is Borel (hence measurable). Finally, we can write a measurable set as the countable union of bounded, measurable sets. Therefore, things boil down to show that the image of a null set under a continuously differentiable map is a null set. It's true because a C^1 function is also locally Lipschitz. By letting the radius of the covering sets be sufficiently small, one can then follow the argument in HW5 Q2.

(You may also follow the proof of Lemma 7.25 by Rudin, Real and Complex Analysis, for a more general case.)

5. Prove or disprove that if $(f_n)_{n=1}^{\infty}$ is a sequence of Lebesgue integrable functions

$$f_n : [0,1] \to \mathbb{R}$$

such that $\lim_{n\to\infty} \int_{[0,1]} |f_n| \, d\mathcal{L} = 0$, then for at least one value $x \in [0,1]$, we have

$$\lim_{n\to\infty} f_n(x) = 0.$$

Solution. The statement is false. For a specific counter example, one can follow the construction in HW3 Q2(a). Let $f_1 = \chi_{[0,1/2]}, f_2 = \chi_{[1/2,1]}, f_3 = \chi_{[0,1/2^2]}, f_4 = \chi_{[1/2^2,1/2]}, f_5 = \chi_{[1/2,3/2^2]}, f_6 = \chi_{[3/2^2,1]}, \cdots$.

It's clear that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n| \, d\mathcal{L} = \lim_{k\to\infty} \frac{1}{2^k} = 0$$

But for any point $x \in [0,1]$, $f_n(x)$ doesn't converge.