Suggested Solution 7

(1) Provide two proofs that $C[0,1]$ is an infinite dimensional vector space.

Solution:

First proof. It is clear that $\{x^n : n = 0, 1, \dots\}$ forms a basis for the subspace $P[0,1] \subset C[0,1]$ of polynomials on $[0,1]$. Hence $\dim C[0,1] \geq \dim P[0,1] = \infty$.

Second proof. Pick continuous function f_n with support inside $(1/(n+1), 1/n)$. Clearly f_n's are linearly independent. The subspace they form are of infinite dimension already.

(2) Show that both $C_c(0,1)$ and $C^1(0,1)$ are not closed subspaces in $C[0,1]$ and hence they are not Banach space.

Solution: For $C_c(0,1)$, we consider $f_n, n > 1$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (0, \frac{1}{2n}), \\ \text{linear} & \text{if } x \in [\frac{1}{2n}, \frac{1}{n}), \\ \frac{1}{2} - |x - \frac{1}{2}| & \text{if } x \in [\frac{1}{n}, 1 - \frac{1}{n}), \\ \text{linear} & \text{if } x \in [1 - \frac{1}{n}, 1 - \frac{1}{2n}), \\ 0 & \text{if } x \in [1 - \frac{1}{2n}, 1). \end{cases}$$

Obviously, $f_n \to \frac{1}{2} - |x - \frac{1}{2}|$ uniformly on $(0,1)$ and hence $C_c(0,1)$ is not closed.

For $C^1(0,1)$, we consider the following example:

$$f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}.$$

with

$$f_n'(x) = \frac{(x - \frac{1}{2})}{\sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}}.$$

and both are continuous on $[0,1]$. Obviously $f_n(x) \to f(x) := |(x - 1/2)|$ pointwisely with the limit does not belong to $C^1[0,1]$. Moreover

$$\begin{aligned} |f - f_n(x)| &= |\sqrt{(x-\frac{1}{2})^2+\frac{1}{n}} - \sqrt{(x-\frac{1}{2})^2}| \\ &= |\frac{\frac{1}{n}}{\sqrt{(x-\frac{1}{2})^2+\frac{1}{n}} + \sqrt{(x-\frac{1}{2})^2}}| \\ &\leq \frac{\frac{1}{n}}{\sqrt{(x-\frac{1}{2})^2+\frac{1}{n}}} \\ &\leq \frac{1}{\sqrt{n}} \end{aligned}$$

Hence $\|f - f_n\|_\infty \to 0$. and the $C^1(0,1)$ is not closed

(3) Endow $C[0,1]$ with the norm $\|f\| = \int_0^1 |f(x)|dx$. Determine whether it is complete or not.

Solution: The space is not complete, we consider $f_n, n > 1$,

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2} - \frac{1}{n}), \\ \text{linear} & \text{if } x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}], \\ -1 & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

$$\forall m > n, \|f_m - f_n\| < \frac{2}{n} \to 0.$$

Then$\{f_n\}$ is Cauchy and obviously there is no $f \in C[0,1]$ s.t.$\|f - f_n\| \to 0$.

(4) Let $C_0(X)$ be the space of all continuous functions vanishing at infinity where X be a locally compact Hausdorff space under the supernorm. A function is called vanishing at infinity if for each $\varepsilon > 0$, there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. Show that $C_0(X)$ is the completion of $C_c(X)$. In other words, $C_0(X)$ is complete and $\overline{C_c(X)} = C_0(X)$.

Solution. Let $C_b(X)$ be the space of all bounded, continuous functions in X. It is undergraduate thing to show $C_b(X)$ is complete under the sup-norm. (You may have done it when X is $\mathbb{R}$, but the proof is the same.) Using $C_0(X) \subset C_b(X)$, it suffices to show that $C_0(X)$ is closed. Let $\{f_n\}$ be a sequence in $C_0(X)$ converging to some $f \in C_b(X)$. For $\varepsilon > 0$, $|f_n(x) - f(x)| \leq \|f_n - f\|_\infty < \varepsilon/2$ for all x and $n \geq n_0$. As $f_{n_0} \in C_0$, there exists a compact set K such that $f_{n_0} < \varepsilon/2$ outside K. It follows that $|f| \leq \|f_{n_0} - f\|_\infty + |f_{n_0}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ outside K. To show that C_c is dense in C_0, let $f \in C_0$. For $\varepsilon > 0$, there is a compact K

such that $|f| < \varepsilon/2$ outside K. Let $\varphi \in C_c(X)$ with $K < \varphi$ and set $g = \varphi f$. Then $f = g$ in K and $|f - g| < |f| + |g| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

(5) Show that the space of bounded sequences, ℓ^∞, is not separable. Hint: Consider all sequences of the form $a = (a_1, a_2, \cdots)$ where $a_j \in \{0, 1\}$.

Solution. There are uncountably many sequences of the form suggested in the hint (they can be put in one-to-one correspondence with the real numbers in $[0, 1]$ via binary representation.) And each pair have distance one. Every dense set will have nonempty intersection with each ball and so it is uncountable.

More precisely, consider a dense set S in ℓ^∞, for each $a \in \{0,1\}^{\mathbb{N}}$, there exists $s_a \in S$ such that $\|s_a - a\|_\infty < 0.5$. Consider the function $f : \{0,1\}^{\mathbb{N}} \to S$ defined by $f(a) := s_a$. If $f(a) = f(b)$ for some $a, b \in \{0,1\}^{\mathbb{N}}$, then $\|a-b\|_\infty \le \|a-f(a)\|_\infty + \|f(b)-b\|_\infty < 1$, whence $a = b$ and f is injective. This shows card $\left(\{0,1\}^{\mathbb{N}}\right) \le \mathrm{card}(S)$

(6) Let Λ be a bounded linear functional on the normed space X. Show that its operator norm

$$\begin{aligned}\|\Lambda\|_{op} &= \sup\left\{\frac{\Lambda x}{\|x\|} : x \neq 0\right\} \\ &= \inf\{M : |\Lambda x| \le M\|x\|,\ \forall x \in X\}.\end{aligned}$$

Solution: To prove the first equality, note that

$$\|\Lambda\|_{op} = \sup\left\{\max\left(\frac{\Lambda x}{\|x\|}, \frac{\Lambda(-x)}{\|-x\|}\right) : x \neq 0\right\} = \sup\left\{\frac{\Lambda x}{\|x\|} : x \neq 0\right\}.$$

For the second, we have $|\Lambda x| \le \|\Lambda\|_{op}\|x\|$, which implies

$$\|\Lambda\|_{op} \ge \inf\{M : |\Lambda x| \le M\|x\|,\ \forall x \in X\}.$$

Also, if M has $|\Lambda x| \le M\|x\|$, $\forall x \in X$, then $\dfrac{|\Lambda x|}{\|x\|} \le M$, which gives $\|\Lambda\|_{op} \le M$. Taking inf on both sides, we have

$$\|\Lambda\|_{op} \le \inf\{M : |\Lambda x| \le M\|x\|,\ \forall x \in X\}.$$

(7) Show that a linear functional in a normed space is bounded if and only if its kernel is closed.

Solution. Let Λ be a bounded functional. The $\ker\lambda = \Lambda^{-1}(\{0\})$ is closed by continuity. On the other hand, if y_n satisfies $\|y_n\| \le M$ but $|\Lambda y_n| \ge n$. The vectors $x_n = y_n/\Lambda y_n$

satisfies $\Lambda x_n = 1$, so $\Lambda(x_n - x_1) = 0$ for all n. But $\lim_{n\to\infty}(x_n - x_1) = -x_1$, but $\Lambda x_1 \neq 0$.

(8) For any normed space $(X, \|.\|)$, prove that $(X', \|.\|_{op})$ forms a Banach space.

Solution. It is clear that X' is a vector space and $\|\cdot\|_{op}$ is a norm on X'. It suffices to prove the completeness.

Suppose $\{\Lambda_n\}$ is Cauchy in $(X', \|\cdot\|_{op})$, i.e.

$$\forall \varepsilon > 0,\ \exists N \text{ such that } \forall m, n \geq N,\ \|\Lambda_m - \Lambda_n\|_{op} < \varepsilon.$$

For any $x \in X$, the inequality $\|\Lambda_m x - \Lambda_n x\| \leq \|\Lambda_m - \Lambda_n\|_{op}\|x\|$ shows that $\{\Lambda_n x\}$ is Cauchy in the scalar field, hence convergent. Define Λ by $\Lambda x = \lim_{n\to\infty} \Lambda_n x$. It is straightforward to verify that Λ is bounded, linear and, in view of

$$\|\Lambda_n - \Lambda\|_{op} = \sup_{\|x\|=1} |\Lambda_n x - \Lambda x| \to 0 \quad \text{as } n \to \infty,$$

we conclude that the dual space of a normed space is always complete. The completeness is in fact inherited from the completeness of the scalar field $\mathbb{R}$.

(9) Consider $C[-1, 1]$ under the sup-norm. Let

$$X = \left\{ f \in C[-1,1] : \int_{-1}^{0} f = \int_{0}^{1} f = 0 \right\},$$

and $g \in C[-1, 1]$ satisfy $\int_0^1 g = 1$, $\int_{-1}^0 g = -1$. Establish the followings:

(a) X is a closed subspace of $C[-1, 1]$.

(b) $\|g - f\|_\infty > 1$, $\forall f \in X$.

(c) $\text{dist}(g, X) = 1$.

Hint: $\int_0^1 (g - f) = 0$ if and only if $g \equiv f$. This example shows that the projection property does not hold in $(C[-1, 1], \|\cdot\|_\infty)$.

Solution. (a) is straightforward. For (b), we observe

$$\int_0^1 (g - f) = 1, \quad \forall f \in X .$$

Therefore, by continuity either $\max_{x\in[0,1]}(g - f)(x) > 1$ or $g - f \equiv 1$ on $[0, 1]$. Similarly,

$$\int_{-1}^{0} (g - f) = -1$$

implies either $\min_{x\in[-1,0]}(g-f)(x) < -1$ or $g - f \equiv -1$ on $[-1,0]$. Since g is continuous, it cannot equal to 1 on $[0,1]$ and -1 on $[-1,0]$, either $\max_{x\in[0,1]}(g-f)(x) > 1$ or $\min_{x\in[-1,0]}(g-f)(x) < -1$ must hold, so $\|g-f\|_\infty > 1$. This is (b).

(c) It suffices to find a sequence $\{f_n\}$ in X such that $\|g-f_n\|_\infty \to 1$. Define a continuous function $h_n : [0,1] \to \mathbb{R}$ by

$$h_n(x) := \begin{cases} 0 & \text{if } x = 0 \\ \text{linear} & \text{if } x \in (0,\delta_n) \\ 1+\frac{1}{n} & \text{if } x \in [\delta_n, 1] \end{cases}$$

where δ_n is a small number such that $\int_0^1 h_n \geq 1 + \frac{1}{2n}$. The function $\Phi : [0,1] \to \mathbb{R}$ defined by $\Phi(\theta) := \int_0^1 (g - \theta h_n)$ is continuous. As $\Phi(0) = 1 > 0$ and $\Phi(1) \leq 1 - \left(1 + \frac{1}{2n}\right) < 0$, by the intermediate value theorem there exists $\theta_0 \in [0,1]$ such that $\Phi(\theta_0) = 0$. If we define $f_n := g - \theta_0 h_n$, then $\int_0^1 f_n = 0$ and $\|g-f_n\|_\infty = \theta_0 \cdot \|h_n\|_\infty \leq 1 + \frac{1}{n}$. We can define $f_n|_{[-1,0]}$ similarly, so that the resulting f_n is continuous on $[-1,1]$ with $f_n(0) = g(0)$, $f_n \in X$, and $\|g-f_n\|_\infty \leq 1+\frac{1}{n}$. Together with (b), we get $\|g-f_n\|_\infty \to 1$.

(10) Let X be a Hilbert space and X_1 a proper closed subspace. For x_0 lying outside X_1, let $d = \|x_0 - z\|$ where d is the distance from x_0 to X_1. Show that

$$\langle x, z - x_0\rangle = 0, \quad \forall x \in X_1.$$

Hint: For $x \in X_1$, one has $\dfrac{d}{dt}\phi(t) = 0$ at $t = 0$ where $\phi(t) = \|z_0 + tx - x_0\|^2$. Why?

Solution. Since $\phi(t)$ attains its minimum at $t = 0$, we have $\phi'(0) = 0$. It is easy to see that

$$\begin{aligned}\phi'(t) &= \frac{d}{dt}\langle z_0 + tx - x_0, z_0 + tx - x_0\rangle \\ &= 2\langle x, z_0 + tx - x_0\rangle.\end{aligned}$$

Putting $t = 0$ yields the result.

(11) Show that the correspondence $\Lambda \mapsto w$ in Theorem 4.8 is norm preserving.

Solution. By Cauchy-Schwarz inequality, $\forall x \in X$,

$$|\Lambda(x)| = |\langle x, w\rangle| \leq \|x\|\|w\|$$

With equality holds when $x = w$. Hence $\|\Lambda\|_{op} = \|w\|$ and the map is norm preserving.

(12) Let Λ_1 and Λ_2 be two bounded linear functionals on the Hilbert space X. Suppose that they have the same kernel. Prove that there exists a nonzero constant c such that $\Lambda_2 = c\Lambda_1$. Use this fact to give a proof of Theorem 4.8

Solution. We may suppose kernel of Λ_1 and Λ_2 is a proper subspace of X and $\exists x_0 \in X, \Lambda_1(x_0), \Lambda_2(x_0) \neq 0$, then $\forall x \in X$,

$$\begin{aligned} \Lambda_1\big(x - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}x_0\big) &= \Lambda_1(x) - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}\Lambda_1(x_0) \\ &= 0. \end{aligned}$$

As the two functionals have the same kernel, we have

$$\begin{aligned} \Lambda_2\big(x - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}x_0\big) &= \Lambda_2(x) - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}\Lambda_2(x_0) \\ &= 0. \end{aligned}$$

Hence

$$\Lambda_2 = \frac{\Lambda_2(x_0)}{\Lambda_1(x_0)}\Lambda_1.$$

Now Let Λ be a non zero bounded linear functional on X and x_0 not in $ker\Lambda$, then $\exists z \in ker\Lambda$ s.t.

$$\langle x, x_0 - z\rangle = 0, \forall x \in ker\Lambda.$$

Theorem 4.8 follows by letting $\Lambda(x) = \Lambda_2(x)$ and $\langle x, x_0 - z\rangle = \Lambda_1(x)$.