Suggested Solutions to Exercise 3

Standard notations are in force.

(1) Prove the conclusion of Lebsegue's dominated convergence theorem still holds when the condition "$\{f_k\}$ converges to f a.e." is replaced by the condition " $\{f_k\}$ converges to f in measure".

Solution. Suppose on the contrary that $\int |f_k - f| d\mu$ does not tend to zero. By considering the limit supremum of the sequence, we can find a positive constant M and a subsequence $\{f_{n_j}\}$ such that

$$\int |f_{n_j} - f| d\mu \geq M$$

for all j. By Prop 1.17, a subsequence $\{g_k\}$ of $\{f_{n_j}\}$ converges to f a.e.. By Lebsegue's dominated convergence theorem, we have

$$0 = \lim_{k\to\infty} \int |g_k - f| d\mu \geq M > 0,$$

contradiction holds.

(2) Find an example in each of the following cases.

(a) A sequence which converges in measure but not at every point.

(b) A sequence which converges pointwisely but not in measure.

(c) A sequence which converges in measure but not in L^1.

Solution.

(a) Take $X = [0,1]$ and the Lebesgue measure. Basically, let $f_1 = \chi_{[0,1/2]}$, $f_2 = \chi_{[1/2,1]}$, $f_3 = \chi_{[0,1/2^2]}$, $f_4 = \chi_{[1/2^2,1/2]}$, $f_5 = \chi_{[1/2,3/2^2]}$, $f_6 = \chi_{[3/2^2,1]}, \cdots$.

(b) According to Proposition 1.18, the measure cannot be finite. Take $X = \mathbb{R}$ and the Lebesgue measure. Consider $g_n = \chi_{[n,n+1]}$. We have $g_n(x) \to 0$ for all x but not in measure.

(c) Take $X = \mathbb{R}$ and the Lebesgue measure. Consider $h_n = n^2\chi_{[0,1/n]}$. Then $h_n \to 0$ in measure but not in L^1.

(3) Let $f_n, n \geq 1$, and f be real-valued measurable functions in a finite measure space. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a subsubsequence that converges to f a.e..

Solution. Let μ be the measure in a finite measure space. If $\{f_n\}$ converges to f in measure, then every subsequence $\{f_{n_k}\}$ also converges to f in measure. By Prop 1.17, the subsequence $\{f_{n_k}\}$ has a sub-subsequence converging to f a.e..

Now suppose that each subsequence of $\{f_n\}$ has a sub-subsequence that converges to f a.e.. Assume that f_n does not converge to f in measure. By considering the limit supremum, there are positive ρ , M and subsequence $\{f_{n_j}\}$ such that, $\mu(\{x : |f_{n_j}(x) - f(x)| \geq \rho\}) \geq M$, for all j. However, a subsequence $\{g_k\}$ of $\{f_{n_j}\}$ converges to f a.e. and by Prop 1.18, $\{g_k\}$ converges to f in measure and

$$0 = \lim_{k\to\infty} \mu(\{x : |g_k(x) - f(x)| \geq \rho\}) \geq M > 0,$$

which is impossible. Hence $\{f_n\}$ converges to f in measure.

(4) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\widetilde{\mathcal{M}}$ contain all sets E such that there exist $A, B \in \mathcal{M}$, $A \subset E \subset B$, $\mu(B\backslash A) = 0$. Show that $\widetilde{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$ and if we set $\widetilde{\mu}(E) = \mu(A)$, then $(X, \widetilde{\mathcal{M}}, \widetilde{\mu})$ is a complete measure space.

Solution. We see that $\widetilde{\mathcal{M}}$ contains $\mathcal{M}$ by taking $E = A = B$ for any $E \in \mathcal{M}$. Suppose $E_i \in \widetilde{\mathcal{M}}$, $B_i \subseteq E_i \subseteq A_i$ where $B_i, A_i \in \mathcal{M}$ and $\mu(A_i\backslash B_i) = 0$, then

$$\bigcap_{i=1}^{\infty} B_i \subseteq \bigcap_{i=1}^{\infty} E_i \subseteq \bigcap_{i=1}^{\infty} A_i$$

and

$$\mu(\bigcap_{i=1}^{\infty} A_i \backslash \bigcap_{i=1}^{\infty} B_i) \leq \mu(\bigcup_{i=1}^{\infty} A_i \backslash B_i) \leq \sum_{i=1}^{\infty} \mu(A_i \backslash B_i) = 0.$$

We have $\bigcap_{i=1}^{\infty} E_i$ is in $\widetilde{\mathcal{M}}$. If $A \supseteq E \supseteq B$, then

$$X \backslash A \subseteq X \backslash E \subseteq X \backslash B$$

and

$$\mu((X \backslash B) \backslash (X \backslash A)) = \mu(A \backslash B).$$

Hence $X \backslash E$ is in $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ is a σ algebra. We check that $\widetilde{\mu}$ is a measure on $\widetilde{M}$. Obviously $\widetilde{\mu}(\phi) = 0$. Let E_i be mutually disjoint $\widetilde{\mu}$ measurable set, $\exists B_i, A_i$ μ measurable s.t

$$A_i \subseteq E_i \subseteq B_i$$

and

$$\mu(B_i \setminus A_i) = 0.$$

Using above argument, we have $\mu(\bigcup_{i=1}^{\infty} B_i \setminus \bigcup_{i=1}^{\infty} A_i) = 0$, And A_i are mutually disjoint,

$$\widetilde{\mu}(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \widetilde{\mu}(E_i).$$

So $\widetilde{\mu}$ is a measure on $\widetilde{M}$.

Finally, we check that $\widetilde{\mu}$ is a complete measure, let E be a $\widetilde{\mu}$ measurable and null set, for all subset $C \subseteq E$, we have $\exists A, B \in \mathcal{M}$ s.t.$A \subseteq E \subseteq B$ and

$\mu(A) = \mu(B) = 0$. Therefore

$$\phi \subseteq C \subseteq B$$

and

$$\mu(B) = 0.$$

We have $C \in \widetilde{\mathcal{M}}$.

(5) Show that $\widetilde{\mathcal{M}}$ in the previous problem is the σ-algebra generated by $\mathcal{M}$ and all subsets of measure zero sets in $\mathcal{M}$.

Solution. Let $\mathcal{M}_1$ be the σ-algebra generated by $\mathcal{M}$ and all subsets of measure zeros sets in $\mathcal{M}$.

By definition, $\widetilde{\mathcal{M}}$ contains all the sets in $\mathcal{M}$ and all subsets of measure zero sets in $\mathcal{M}$. Since $\mathcal{M}_1$ is the smallest such σ-algebra, we have $\mathcal{M}_1 \subset \overline{\mathcal{M}}$.

To prove that $\widetilde{\mathcal{M}} \subset \mathcal{N}$, we let $E \in \widetilde{\mathcal{M}}$. Then there exist $A, B \in \mathcal{M}$ such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Then $E \setminus A \subset B \setminus A$ is a subset of a measure zero set. Now $E = A \cup (E \setminus A)$ is a union of a set in $\mathcal{M}$ and a subset of a measure zero set. Hence, $E \in \mathcal{M}_1$.

(6) Here we consider an application of Caratheodory's construction. An *algebra* $\mathcal{A}$ on a set X is a subset of $\mathcal{P}_X$ that contains the empty set and is closed under taking complement and finite union. A *premeasure* $\mu : \mathcal{A} \to [0, \infty]$ is a finitely additive function which satisfies: $\mu(\phi) = 0$ and $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$ whenever E_k are disjoint and $\bigcup_{k=1}^{\infty} E_k \in \mathcal{A}$. Show that the premeasure μ can be extended to a measure on the σ-algebra generated by $\mathcal{A}$. Hint: Define the outer measure

$$\overline{\mu}(E) = \inf \Big\{ \sum_k \mu(E_k) : E \subset \bigcup_k E_k, E_k \in \mathcal{A} \Big\}.$$

This is called Hahn-Kolmogorov theorem.

Solution. We follow the proof from Terence Tao's book "Introduction To Measure Theory". Define $\overline{\mu}$ as above and obviously $\overline{\mu}$ is an outer measure on power set of X. By Caratheodory's construction, we get a measure defined on a σ- algebra M. We claim that $\mathcal{A} \subseteq M$, let $E \in \mathcal{A}$ and $C \subseteq X$ such that $\overline{\mu}(C) < \infty$, for all $\varepsilon > 0$, there is $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ covering C such that

$$\overline{\mu}(C) + \varepsilon > \sum_{i=1}^{\infty} \mu(E_i).$$

As $\{E_i \cap E\}_{i=1}^{\infty}$ and $\{E_i \setminus E\}_{i=1}^{\infty}$ are subset of $\mathcal{A}$ and cover $C \cap E$ and $C \setminus E$ respectively, we have,

$$\overline{\mu}(C \cap E) \leq \sum_{i=1}^{\infty} \mu(E_i \cap E),$$

and

$$\overline{\mu}(C \setminus E) \leq \sum_{i=1}^{\infty} \mu(E_i \setminus E).$$

Using the fact that μ is a premeasure, $\mu(E_i \cap E) + \mu(E_i \setminus E) = \mu(E_i)$. Summing over i, we know that E is in M and M contains the σ algebra generated by $\mathcal{A}$. Now we try to show that the measure induced extends μ, obviously by definition $\overline{\mu}(E) \leq \mu(E)$. Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ covering E. Without affecting the countable union, we may make $\{E_i\}_{i=1}^{\infty}$ disjoint and obtain $\{B_i\}_{i=1}^{\infty}$. Furthermore, by taking intersection with E, we have

$$\bigcup_{i=1}^{\infty} B_i \cap E = E$$

and

$$\sum_{i=1}^{\infty} \mu(E_i) \geq \sum_{i=1}^{\infty} \mu(B_i) \geq \sum_{i=1}^{\infty} \mu(B_i \cap E) = \mu(E),$$

where the last equality follows from the condition of μ. Hence $\overline{\mu}(E) \geq \mu(E)$

and the measure extends μ.

The following problems are concerned with the Lebesgue measure. Let $R = I_1 \times I_2 \times \cdots \times I_n$, I_j bounded intervals (open, closed or neither), be a rectangle in $\mathbb{R}^n$. More properties of the Lebesgue measure can be found in Exercise 4.

(7) For a rectangle R in $\mathbb{R}^n$, define its "volume" to be

$$|R| = (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_n - a_n)$$

where b_i, a_i are the right and left endpoints of I_j. Show that

(a) if $R = \bigcup_{k=1}^{N} R_k$ where R_k are almost disjoint, then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

(b) If $R \subset \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Solution.

(a) Take $n = 2$ for simplicity. Each rectangle is of the form $[a, b] \times [c, d]$. We order the endpoints of the x-coordinates of all rectangles $R_1, \cdots, R_N$ into $a_1 < a_2 < \cdots < a_n$ and y-coordinates into $b_1 < b_2 < \cdots < b_m$. This division breaks R into an almost disjoint union of subrectangles $R_{j,k} = [a_j, a_{j+1}] \times [b_k, b_{k+1}]$. Note that each $R_{j,k}$ is contained in exactly one R_l and each R_l is an almost disjoint union of subrectangles from this division. We have

$$\begin{aligned}
|\mathbb{R}| &= (a_n - a_1)(b_m - b_1) \\
&= (a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1)(b_m - b_{m-1} + b_{m-1} + \cdots + b_2 - b_1) \\
&= \sum_{j,k} |R_{j,k}| \\
&= \sum_{l} \sum_{R_{j,k} \subseteq R_l} |R_{j,k}| \\
&= \sum_{l} |R_l|,
\end{aligned}$$

(b) The proof is similar to that of (a), but now we need to subdivide R_j and R together. Now, $R \subseteq \bigcup_{j=1}^{N} R_j$. We order all x-coordinates of R_j, R into $a_1 < a_2 < \cdots < a_N$ and and y-coordinates into $b_1 < b_2 < \cdots < b_M$. Then R is the union of parts of $R_{k,j}$

$$\begin{aligned}
|R| &= \sum_{R_{j,k} \subseteq R} |R_{j,k}| \\
&\le \sum_{j,k} |R_{j,k}| \\
&\le \sum_{j} |R_j|,
\end{aligned}$$

where the last inequality follows from the fact that each $R_{k,j}$ is contained in some R_j.

(8) Let $\mathcal{R}$ be the collection of all closed cubes in $\mathbb{R}^n$. A closed cube is of the form $I \times \cdots \times I$ where I is a closed, bounded interval.

(a) Show that $(\mathcal{R}, |\cdot|)$ forms a gauge, and thus it determines a complete measure $\mathcal{L}^n$ on $\mathbb{R}^n$ called the *Lebesgue measure*.

(b) $\mathcal{L}^n(R) = |R|$ where R is a cube, closed or open.

(c) For any set E and $x \in \mathbb{R}^n$, $\mathcal{L}^n(E+x) = \mathcal{L}^n(E)$. Thus the Lebsegue measure is translational invariant.

Solution.

(a) We clearly have

$$\inf_{R\in\mathcal{R}} |R| = 0 \quad \text{and} \quad \bigcup_{R\in\mathcal{R}} R = \mathbb{R}^n.$$

Hence, $\mathcal{R}$ forms a gauge.

Define μ by

$$\mu(A) = \inf\left\{\sum_{j=1}^{\infty} |G_j| : G_j \in \mathcal{R}\right\}.$$

We check that μ is an outer measure.

- Clearly, $\mu(\phi) = 0$.
- Suppose $\{E_j : j \in \mathcal{N}\}$ are given and write $E = \bigcup_{j\in\mathbb{N}} E_j$. Let $\varepsilon > 0$. Choose $G_{jk} \in \mathcal{R}$ such that

  $$\mu(E_j) + 2^{-j}\varepsilon > \sum_{k\in\mathbb{N}} |G_{jk}|.$$

  Then $\{G_{jk} : j,k \in \mathbb{N}\}$ is a countable cover for E. We have

  $$\mu(E) \le \sum_{j,k\in\mathbb{N}} |G_{jk}| < \sum_{j\in\mathbb{N}} \left(\mu(E_j) + 2^{-j}\varepsilon\right) = \sum_{j\in\mathbb{N}} \mu(E_j) + \varepsilon.$$

  Taking $\varepsilon \to 0$, we have

  $$\mu(E) \le \sum_{j\in\mathbb{N}} \mu(E_j).$$

Following the Carathéodory's construction, we obtain a complete measure.

(b) By (9)(b), it suffices to show that $R \subseteq \bigcup^{\infty} R_j \Rightarrow |R| \leq \sum^{\infty} |R_j|$. We replace $R_j = [a_j, b_j] \times [c_j, d_j]$ by $\acute{R_j} = (a_j - \frac{\varepsilon}{2^j}, b_j + \frac{\varepsilon}{2^j}) \times (c_j - \frac{\varepsilon}{2}, d_j + \frac{\varepsilon}{2})$. Since $\{\acute{R_j}\}$ is an open cover of R and R is compact, there exists a finite subcover $\acute{R}_{j_1}, \cdots, \acute{R}_{j_M}$. By (9)(b)

$$|R| \leq \sum_{k=1}^{M} |\acute{R}_{j_k}| \leq \sum_{j,k} |R_{j,k}| \leq \sum_{j} |R_j| + C\varepsilon,$$

C depends on n only. Let $\varepsilon \to 0$,

$$|R| \leq \sum_{j} |R_j|,$$

which shows that

$$|R| = \inf\{\sum_{1}^{\infty} |R_j| : R \subseteq \bigcup_{j=1}^{\infty} R_j, R_j \text{ closed cube}\}.$$

Therefore,

$$\mathcal{L}^n(R) = |R|$$

where R is a closed cube.

In order to show it also holds for any cube, it suffices to show $\mathcal{L}^n(F) = 0$ whenever F is a face of R. First, $\mathcal{L}^n(F) \leq \mathcal{L}^n(R) < \infty$. Let N be any number> 1 and $x_1 = (a_1, 0), \cdots, x_N = (a_N, 0)$ be distinct point. Consider $F + x_j$. For small a_j, $F + x_j$ can be chosen to sit inside R, then $\bigcup(F + x_j) \subset R$. By (c), $N\mathcal{L}^n(F) = \sum \mathcal{L}^n(F + x_j) = L^n(\bigcup(F + x_j)) \leq \mathcal{L}^n(R) \Rightarrow \mathcal{L}^n(F) \leq \dfrac{\mathcal{L}^n(R)}{N} \to 0$ as $N \to \infty$.

(c) Result follows directly from definition.

(9) This problem is optional. Use Hahn-Kolmogorov theorem to construct the Lebesgue measure instead of Problems 7 and 8.

Solution. Refer to section 20.2 in Royden's book, *Real Analysis.*