# Fall 2022
# MATH5011 Suggested Solution to Exercise 2

(1) Let $g$ be a measurable function in $[0, \infty]$. Show that

$$m(E) = \int_E g \, d\mu$$

defines a measure on $\mathcal{M}$. Moreover,

$$\int_X f \, dm = \int_X fg \, d\mu, \qquad \forall f \text{ measurable in } [0, \infty].$$

**Solution**: We readily check that

(1) $m(\phi) = 0$;

(2) $m(E) \geq 0$, $\forall E \in M$;

(3) For mutually disjoint $A_k \in M$,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \int_X \sum_{k=1}^{\infty} \chi_{A_k} g \, d\mu = \sum_{k=1}^{\infty} \int \chi_{A_k} g \, d\mu = \sum_{k=1}^{\infty} m(A_k)$$

by monotone convergence theorem, since $\sum_{k=1}^{n} \chi_{A_k} g \uparrow \sum_{k=1}^{\infty} \chi_{A_k} g$.

To prove the last assertion, consider the following cases:

(a) $f = \chi_E$ for some $E \in M$.

$$\int_X f \, dm = \int_E dm = m(E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu = \int_X fg \, d\mu.$$

(b) $f$ is a non-negative simple function.

This follows from (a).

(c) $f$ is a non-negative measurable function.

Pick a sequence $s_n \geq 0$ of simple functions such that $s_n \uparrow f$ pointwisely. Then $0 \leq s_n g \uparrow g$ pointwisely. From (b),

$$\int_X s_n \, dm = \int_X s_n g \, d\mu.$$

Taking $n \to \infty$, by monotone convergence theorem, we have

$$\int_X f \, dm = \int_X f g \, d\mu.$$

(2) Let $\{f_k\}$ be measurable in $[0, \infty]$ and $f_k \downarrow f$ a.e., $f$ measurable and $\int f_1 \, d\mu < \infty$. Show that

$$\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu.$$

What happens if $\int f_1 \, d\mu = \infty$?

**Solution**: From the assumption we know the integrability of $f_1$ implies that all $f_k$ are integrable. Without loss of generality, we may suppose $f_k \downarrow f$ pointwisely. (Otherwise, replace by $X$ by $Y = X \setminus N$, such that $\mu(N) = 0$ and $f_k \downarrow f$ on $Y$.) Then $0 \leq f_1 - f_k \uparrow f_1 - f$. By monotone convergence theorem,

$$\lim_{k \to \infty} \int_X (f_1 - f_k) \, d\mu = \int_X (f_1 - f) \, d\mu.$$

Since $\int_X f_1 \, d\mu < \infty$, we can cancel it from both sides to yield the result.

If $\int_X f_1 \, d\mu = \infty$, the result does not hold. For example, one may take $X = \mathbb{R}$, $f_k(x) = 1/k$ and $f = 0$. Then

$$\int_X f \, d\mu = 0, \text{ while } \int_X f_k \, d\mu = \infty, \ \forall k \in N.$$

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(3) Let $f$ be a measurable function. Show that there exists a sequence of simple functions $\{s_j\}$, $|s_1| \leq |s_2| \leq |s_3| \leq \cdots$, and $s_k(x) \to f(x)$, $\forall x \in X$.

**Solution**: Choose sequences of non-negative simple functions $s_j^+ \uparrow f_+$ and $s_j^- \uparrow f_-$. Put $s_j = s_j^+ \chi_{\{x:f(x)\geq 0\}} - s_j^- \chi_{\{x:f(x)<0\}}$. Fix $x \in X$. If $f(x) \geq 0$ then $|s_j(x)| = s_j^+(x) \uparrow f_+$. If $f(x) < 0$ then $|s_j(x)| = s_j^-(x) \uparrow f_-$. We also have

$$s_j(x) \to f_+ \chi_{\{x:f(x)\geq 0\}}(x) - f_- \chi_{\{x:f(x)<0\}}(x) = f(x), \quad \forall x \in X.$$

(4) Let $\mu(X) < \infty$ and $f$ be integrable. Suppose that

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in [a, b], \ \forall E \in \mathcal{M}, \mu(E) > 0$$

for some $[a, b]$. Show that $f(x) \in [a, b]$ a.e..

**Solution**: Let $A = \{x : f(x) < a\}$ and $B = \{x : f(x) > b\}$. If $\mu(A) > 0$, we will draw a contradiction. Let $A_n = \{x \in A : f(x) < a - 1/n\}$ so $A = \bigcup_n A_n$. As $\{A_n\}$ is an ascending family tending to $A$, we can find some $n_0$ such that $\mu(A_{n_0}) > 0$. Then

$$\frac{1}{\mu(A_{n_0})} \int_{A_{n_0}} f d\mu \leq a - \frac{1}{n_0},$$

contradiction. Similarly we can treat the case $\mu(B) > 0$.

(5) Let $f$ be Lebsegue integrable on $[a, b]$ which satisfies

$$\int_a^c f d\mathcal{L}^1 = 0,$$

for every $c$. Show that $f$ is equal to $0$ a.e..

**Solution**: We can express our assumption as

$$\int_a^c f_+ d\mathcal{L}^1 = \int_a^c f_- d\mathcal{L}^1, \qquad \forall c \in [a, b] \ .$$

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Clearly this implies these two integrals holds when $(a, c)$ is replaced by any open interval. As every open set in $[a, b]$ can be written as a countable disjoint union of open intervals, these two integrals are equal over any open set. From Lebesgue integration theory we know that for every Lebesgue measurable $E$, there is an open set $G$ containing $E$ with the approximating measure. Thus we conclude

$$\int_E f_+ d\mathcal{L}^1 = \int_E f_+ d\mathcal{L}^1,$$

for all measurable $E$. Taking $E = \{x \in [a, b] : f(x) > 0$, we see that $\int_E f_+ d\mathcal{L}^1 = 0$, which implies $f \leq 0$ a.e.. By taking $E = \{x : f(x) < 0\}$, we see that $f \geq 0$ a.e. . Hence $f = 0$ a.e. .

(6) Let $f \geq 0$ be integrable and $\int f \, d\mu = c \in (0, \infty)$. Prove that

$$\lim_{n \to \infty} \int n \log \left(1 + \left(\frac{f}{n}\right)^\alpha\right) d\mu = \begin{cases} \infty, & \text{if } \alpha \in (0, 1) \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

**Solution**: Let $g_n(x) = n \log \left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right)$. Since $\int f \, d\mu = c \in (0, \infty)$, we know that $\mu(\{x : f(x) = \infty\}) = 0$ and $\mu(\{x : f(x) > 0\}) > 0$. Observe that

$$\lim_{n \to \infty} g_n(x) = \begin{cases} \infty, & \text{on } \{x : f(x) > 0\}, \text{ if } \alpha < 1, \\ f(x), & \text{a.e. } \mu, \text{ if } \alpha = 1, \\ 0, & \text{a.e. } \mu, \text{ if } \alpha > 1. \end{cases}$$

Moreover, if $\alpha \geq 1$, using the elementary inequalities $1 + x^\alpha \leq (1 + x)^\alpha$ and $\log(1 + x) \leq x$ for $x \geq 0$, we have

$$g_n \leq n \log \left(1 + \frac{f}{n}\right)^\alpha \leq n\alpha \cdot \frac{f}{n} = \alpha f \in L^1(\mu).$$

- Suppose $\alpha \in (0, 1)$. By Fatou's lemma,

$$\varliminf_{n \to \infty} \int g_n \, d\mu \geq \int \varliminf_{n \to \infty} g_n \, d\mu = \infty.$$

Hence, $\lim\limits_{n \to \infty} \int g_n \, d\mu = \infty$.

- Suppose $\alpha = 1$. By Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int g_n \, d\mu = \int \lim_{n \to \infty} g_n \, d\mu = \int f \, d\mu = c.$$

- Suppose $1 < \alpha < \infty$. By Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int g_n \, d\mu = \int \lim_{n \to \infty} g_n \, d\mu = 0.$$

(7) Let $\mu(X) < \infty$ and $f_k \to f$ uniformly on $X$ and each $f_k$ is bounded. Prove that

$$\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu.$$

Can $\mu(X) < \infty$ be removed?

**Solution**: We assume that $\mu(X) > 0$. (Otherwise, the result is trivial.) Let $\varepsilon > 0$ be given. Since $f_k \to f$ uniformly on $X$, there exists natural number $N$ such that for all $k \geq N$ and for all $x \in X$, we have

$$|f_k(x) - f(x)| < \frac{\varepsilon}{\mu(X)}.$$

So, for all $k \geq N$, we have

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| \leq \int |f_k - f| \, d\mu < \varepsilon.$$

The result follows.

If $\mu(X) = \infty$, the result no longer holds. One may take $X = R$, $f_k(x) = 1/k$,

$f(x) = 0$ and $\mu$ to be the Lebesgue measure. Then $f_k \to f$ uniformly on $X$ and each $f_k$ is bounded,

$$\int f \, d\mu = 0, \text{ while } \int f_k \, d\mu = \infty, \ \forall k.$$

(8) Give another proof of Borel-Cantelli lemma Problem 7 in Ex 1 by integration theory. (Hint: Study $g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x)$.)

**Solution**: Let $\{A_k\}$ be measurable, $A = \{x \in X : x \in A_k \text{ for infinitely many } k\}$ and suppose $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Write

$$g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x).$$

Then $x \in A$ if and only if $g(x) = \infty$. By Fatou's lemma,

$$\int g \, d\mu \leq \sum_{j=1}^{\infty} \int \chi_{A_j} \, d\mu = \sum_{j=1}^{\infty} \mu(A_j) < \infty.$$

As a consequence of Markov's inequality, $g$ is finite a.e., the conclusion follows.

This problem shows the power by expressing things in terms of measurable functions.

(9) Let $f$ be a Riemann integrable function on $[a, b]$ and extend it to $\mathbb{R}$ by setting it zero outside $[a, b]$.

   (a) Show that $f$ is Lebsegue measurable.

   (b) Show that the Riemann integral of $f$ is equal to $\int_{\mathbb{R}} f d\mathcal{L}^1$.

   (c) Give an example of a sequence of Riemann integrable functions which is uniformly bounded on $[a, b]$ and converges pointwisely to some function which is not Riemann integrable.

**Solution**:

(a) We assume the result and notation in question 10 of exercise 1, by the proof of 10b), f is Riemann integrable on $[a, b]$ if and only if $\overline{R}(f) = \underline{R}(f)$. When this holds, $L = \overline{R}(f) = \underline{R}(f)$. Then for all natural number $n$, we may find partition of [a,b], $P_n = \{a = z_0 < z_1 < \ldots < z_{m_n} = b\}$ such that

$$0 \leq \overline{R}(P_n, f) - \underline{R}(P_n, f) \leq \frac{1}{n},$$

define two sequence of step function in the following way, for all x in $[z_j, z_{j+1})$,

$$\varphi_n(x) = \inf \left\{ f(x) : x \in [z_j, z_{j+1}] \right\},$$

and

$$\psi_n(x) = \sup \left\{ f(x) : x \in [z_j, z_{j+1}] \right\}.$$

For all x in $[a, b]$

$$h(x) = \sup \left\{ \varphi_n(x) : n \in N \right\}$$

and

$$g(x) = \inf \left\{ \psi_n(x) : n \in N \right\},$$

$h$ and $g$ are obviously Lebesgue measurable, we also have $\varphi_n(x) \leq h \leq f \leq g \leq \psi_n(x)$. For any natural number $n$,

$$0 \leq \int_a^b (g - h)d\mathcal{L}^1 \leq \int_a^b (\psi_n - \varphi_n)d\mathcal{L}^1 = \overline{R}(P_n, f) - \underline{R}(P_n, f) \leq \frac{1}{n},$$

so we have $h = f = g$ a.e. and $f$ is Lebesgue measurable.

(b) By taking refinement with the partition $\{a = z_0 < z_1 = a + (b - a)/n < \ldots < z_j = a + j(b - a)/n < \ldots < z_{m_n} = b\}$ if necessary, we may assume the norm of partition $P_n$ in (a) tend to 0 as $n \to 0$. As $\varphi_n$ and $\psi_n$ are integrable and $|f(x)| \leq |\varphi_n(x)| + |\psi_n(x)|$ for all $x$ in $[a, b]$, $f$ is Lebesgue

integrable and

$$\underline{R}(P_n, f) = \int_a^b \varphi_n d\mathcal{L}^1 \leq \int_a^b f d\mathcal{L}^1 \leq \int_a^b \psi_n d\mathcal{L}^1 = \overline{R}(P_n, f) \ .$$

Using result in 10(b) of Ex.1 and let n go to $\infty$, we have Riemann integral $= \int_R f d\mathcal{L}^1$.

(c) We consider the famous Dirichlet function g which is not Riemann integrable, $g(x) = 1$ if $x$ is rational and $\in [0, 1]$ , $g(x) = 0$ otherwise. Let $\{q_n : n \in N\}$ be an enumeration of all rational number in $[0, 1]$ and define

$$f_n = \sum_{i=1}^n \chi_{q_i} \ .$$

Then each $f_n$ is obviously uniformly bounded Riemann integrable with zero integral and yet $\{f_n\}$ converges pointwisely to the Dirichlet function for all x in $[0, 1]$.

(10) Let $f$ be integrable in $(X, \mathcal{M}, \mu)$. Show that for each $\varepsilon > 0$, there is some $\delta$ such that

$$\int_E |f| < \varepsilon, \quad \text{whenever } E \in \mathcal{M}, \ \mu(E) < \delta \ .$$

This is called the absolute continuity of an integrable function.

**Solution.** Assume on the contrary there is some $\varepsilon_0 > 0$ and $E_j, \mu(E_n) \leq 2^{-n}$, such that $\int_{E_n} |f| d\mu \geq \varepsilon_0$. Let $A_n = \bigcup_{j \geq n} E_j$. Then

$$\mu(A_n) \leq \sum_{j \geq n} \mu(E_j) \leq \sum_{j \geq n} \frac{1}{2^j} = \frac{1}{2^{n-1}} \ .$$

Let $A = \cap_n A_n$. As $\{A_n\}$ is descending and $\mu(A_1)$ is finite,

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0 \ ,$$

that is, $A$ is of measure zero. On the other hand, we have $|f|\chi_{A_n} \leq |f|$, by

the dominated convergence theorem we have

$$\int_A |f| d\mu = \lim_{n \to \infty} \int_{A_n} |f| d\mu \geq \varepsilon_0 > 0 \ ,$$

contradiction holds.