Fall 2022

MATH5011 Real Analysis I

Exercise 1 Suggested Solution

(1) Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of measurable sets in $(X, \mathcal{M})$. Let

$$A = \{x \in X : \ x \in A_k \text{ for infinitely many k}\} \ ,$$

and

$$B = \{x \in X : x \in A_k \text{ for all except finitely many k}\} \ .$$

Show that A and B are measurable.

Solution

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k.$$

$$B = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k.$$

(2) Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\Psi(f, g)$ are measurable for any measurable functions f, g. This result contains Proposition 1.3 as a special case.

Solution Note that every open set $G \subseteq \mathbb{R}^2$ can be written as a countable union of set of the form $V_1 \times V_2$ where V_1, V_2 open in $\mathbb{R}$. (Think of $V_1 \times V_2 = (a, b) \times (c, d), a, b, c, d \in Q)$.

Let $G \subseteq \mathbb{R}$ be open. Then $\Phi^{-1}(G)$ is open in $\mathbb{R}^2$, so

$$\Phi^{-1}(G) = \bigcup_n (V_n^1 \times V_n^2),$$

Then

$$h^{-1}(\Phi^{-1})(G) = \bigcup_n h^{-1}(V_n^1 \times V_n^2) = \bigcup_n f^{-1}(V_n^1) \cap g^{-1}(V_n^2)$$

is measurable since f and g are measurable. Hence $h = (f, g)$.

(3) Show that $f : X \to \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}([a, b])$ is measurable for all $a, b \in \overline{\mathbb{R}}$.

Solution By def $f : X \to \overline{R}$ is measurable if $f^{-1}(G)$ is measurable. $\forall G$ open in $\overline{R}$. Every open set G in $\overline{R}$ can be written as a countable union of (a, b), $[-\infty, a)$, $(b, \infty]$, $a, b \in R$. So ff is measurable iff $f^{-1}(a, b)$, $f^{-1}[-\infty, a)$, $f^{-1}(b, \infty]$ are measurable.

$\Rightarrow$) Use

$$f^{-1}(a, b) = \bigcap_n f^{-1}\left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$

$$f^{-1}[-\infty, a) = \bigcap_n f^{-1}\left[-\infty, a + \frac{1}{n}\right)$$

$$f^{-1}(b, \infty] = \bigcap_n f^{-1}\left(b - \frac{1}{n}, \infty\right]$$

$\Leftarrow$) Use

$$f^{-1}(a, b) = \bigcup_n f^{-1}\left[a - \frac{1}{n}, b + \frac{1}{n}\right]$$

$$f^{-1}[-\infty, a) = \bigcap_n f^{-1}\left[-\infty, a - \frac{1}{n}\right]$$

$$f^{-1}(b, \infty] = \bigcap_n f^{-1}\left[b + \frac{1}{n}, \infty\right].$$

(4) Let $f : X \times [a, b] \to \mathbb{R}$ satisfy (a) for each x, $y \mapsto f(x, y)$ is Riemann integrable, and (b) for each y, $x \mapsto f(x, y)$ is measurable with respect to some

σ-algebra $\mathcal{M}$ on X. Show that the function

$$F(x) = \int_a^b f(x,y)dy$$

is measurable with respect to $\mathcal{M}$.

Solution For simplicity let $[a,b] = [0,1]$. For $n \geq 1$, equally divide $[0,1]$ into subintervals of length $1/n$ and let

$$F_n(x) = \sum_{k=1}^{n} f\left(x, \frac{k}{n}\right)\frac{1}{n} \ .$$

Clearly F_n is measurable (with respect to $\mathcal{M}$). Now

$$F(x) = \lim_{n\to\infty} F_n(x) \ ,$$

so it is also measurable.

(5) Let f, g, $f_k, k \geq 1$, be measurable functions from X to $\overline{\mathbb{R}}$.

(a) Show that $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable sets.

(b) Show that $\{x : \lim_{k\to\infty} f_k(x)$ exists and is finite$\}$ is measurable.

Solution

(a) Suffice to show $\{x : F(x) > 0\}$ and $\{x : F(x) = 0\}$ are measurable. If F is measurable, use

$$\{x : F(x) > 0\} = F^{-1}(0,\infty]$$

$$\{x : F(x) = 0\} = F^{-1}[0,\infty] \cap F^{-1}[-\infty,0]$$

Alternatively, one may consider

$$\{x \in X : f(x) < g(x)\} = \bigcup_{r\in\mathbb{Q}} \left(f^{-1}[-\infty,r) \cap g^{-1}(r,\infty]\right)$$

and

$$\{x \in X : f(x) = g(x)\} = \{x \in X : f(x) < g(x)\}^c \cap \{x \in X : f(x) > g(x)\}^c$$

(b) Since $g(x) = \limsup_{k\to\infty} f_k(x)$ and $\liminf_{k\to\infty} f_k(x)$ are measurable.

$$\{x : \lim_{k\to\infty} f_k(x) \text{ exists }\} = \{x : \liminf_{k\to\infty} f_k(x) = \limsup_{k\to\infty} f_k(x)\}$$

On the other hand, the set$\{x : g(x) < +\infty\}$ is also measurable, so is their intersection.

(6) There are two conditions (i) and (ii) in the definition of a measure μ on $(X, \mathcal{M})$. Show that (i) can be replaced by the "nontriviality condition": There exists some $E \in \mathcal{M}$ with $\mu(E) < \infty$.

Solution If μ is a measure satisfying the nontriviality condition and (ii), let $A_1 = E$, $A_i = \phi$ for $i \geq 2$ in ii),

$$\infty > \mu(E) = \sum_{i=1}^{\infty} \mu(A_i) \geq \mu(A_1) + \mu(A_2) = \mu(E) + \mu(\phi)$$

so $0 \geq \mu(\phi) \geq 0$. We have μ is a measure satisfying (i) and (ii).
If μ is a measure satisfying (i) and (ii), taking $E = \phi$, we have the nontriviality condition.

(7) Let $\{A_k\}$ be measurable and $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and

$$A = \{x \in X : x \in A_k \text{ for infinitely many } k\}.$$

We know that A is measurable from (1). Show that $\mu(A) = 0$.

Solution Since $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, we have $\sum_{k=n}^{\infty} \mu(A_k) \to 0$ as $n \to \infty$. For any n$\in N$, we have

$$A \subset \bigcup_{k \geq n} A_k$$

and so

$$\mu(A) \leq \sum_{k=n}^{\infty} \mu(A_k) \ .$$

Taking $n \to \infty$, we have $\mu(A) = 0$.

This result is called Borel-Cantelli lemma.

(8) Let B be the set defined in (1). Let μ be a measure on $(X, \mathcal{M})$. Show that

$$\mu(B) \leq \liminf_{k \to \infty} \mu(A_k) \ .$$

Solution Using the characterization

$$B = \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} A_j \ ,$$

and the fact that $\{\cap_{j \geq k} A_j\}$ is ascending in k, we have

$$\begin{aligned} \mu(B) &= \lim_{k \to \infty} \mu \left(\bigcap_{j \geq k} A_j \right) \\ &= \liminf_{k \to \infty} \mu \left(\bigcap_{j \geq k} A_j \right) \\ &\leq \liminf_{k \to \infty} \mu(A_k) \ . \end{aligned}$$

(9) Here we review Riemann integral. Let f be a bounded function defined on $[a, b], a, b \in \mathbb{R}$. Given any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ on $[a, b]$ and tags $z_j \in [x_j, x_{j+1}]$, there corresponds a *Riemann sum* of f given by $R(f, P, \mathbf{z}) = \sum_{j=0}^{n-1} f(z_j)(x_{j+1} - x_j)$. The function f is called *Riemann integrable*

with integral L if for every $\varepsilon > 0$ there exists some δ such that

$$\left|R(f, P, \mathbf{z}) - L\right| < \varepsilon,$$

whenever $\|P\| < \delta$ and $\mathbf{z}$ is any tag on P. (Here $\|P\| = \max_{j=0}^{n-1} |x_{j+1} - x_j|$ is the length of the partition.) Show that

1. For any partition P, define its *Darboux upper* and *lower sums* by

   $$\overline{R}(f, P) = \sum_j \sup\left\{f(x) : x \in [x_j, x_{j+1}]\right\}(x_{j+1} - x_j),$$

   and

   $$\underline{R}(f, P) = \sum_j \inf\left\{f(x) : x \in [x_j, x_{j+1}]\right\}(x_{j+1} - x_j)$$

   respectively. Show that for any sequence of partitions $\{P_n\}$ satisfying $\|P_n\| \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \overline{R}(f, P_n)$ and $\lim_{n\to\infty} \underline{R}(f, P_n)$ exist.

2. $\{P_n\}$ as above. Show that f is Riemann integrable if and only if

   $$\lim_{n\to\infty} \overline{R}(f, P_n) = \lim_{n\to\infty} \underline{R}(f, P_n) = L.$$

3. A set E in $[a, b]$ is called *of measure zero* if for every $\varepsilon > 0$, there exists a countable subintervals J_n satisfying $\sum_n |J_n| < \varepsilon$ such that $E \subset \bigcup_n J_n$. Prove Lebsegue's theorem which asserts that f is Riemann integrable if and only if the set consisting of all discontinuity points of f is a set of measure zero. Google for help if necessary.

Solution:

(a) It suffices to show: For every $\varepsilon > 0$, there exists some δ such that

$$0 \le \overline{R}(f, P) - \overline{R}(f) < \varepsilon,$$

and

$$0 \leq \underline{R}(f) - \underline{R}(f, P) < \varepsilon,$$

for any partition P, $\|P\| < \delta$, where

$$\overline{R}(f) = \inf_P \overline{R}(f, P),$$

and

$$\underline{R}(f) = \sup_P \underline{R}(f, P).$$

.

If it is true, then $\lim_{n\to\infty} \overline{R}(f, P_n)$ and $\lim_{n\to\infty} \underline{R}(f, P_n)$ exist and equal to $\overline{R}(f)$ and $\underline{R}(f)$ respectively.

Given $\varepsilon > 0$, there exists a partition Q such that

$$\overline{R}(f) + \varepsilon/2 > \overline{R}(f, Q).$$

Let m be the number of partition points of Q (excluding the endpoints). Consider any partition P and let R be the partition by putting together P and Q. Note that the number of subintervals in P which contain some partition points of Q in its interior must be less than or equal to m. Denote the indices of the collection of these subintervals in P by J. We have

$$0 \leq \overline{R}(f, P) - \overline{R}(f, R) \leq \sum_{j \in J} 2M \Delta x_j \leq 2M \times m||P||,$$

where $M = \sup_{[a,b]} |f|$, because the contributions of $\overline{R}(f, P)$ and $\overline{R}(f, Q)$ from the subintervals not in J cancel out. Hence, by the fact that R is a refinement of Q,

$$\overline{R}(f) + \varepsilon/2 > \overline{R}(f, Q) \geq \overline{R}(f, R) \geq \overline{R}(f, P) - 2Mm||P||,$$

i.e.,

$$0 \le \overline{R}(f,P) - \overline{R}(f) < \varepsilon/2 + 2Mm||P||.$$

Now, we choose

$$\delta < \frac{\varepsilon}{1+4Mm},$$

Then for P, $\|P\| < \delta$,

$$0 \le \overline{R}(f,P) - \overline{R}(f) < \varepsilon.$$

Similarly, one can prove the second inequality.

(b) With the result in part a, it suffices to prove the following result: Let f be bounded on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if $\overline{R}(f) = \underline{R}(f)$. When this holds, $L = \overline{R}(f) = \underline{R}(f)$.

According to the definition of integrability, when f is integrable, there exists some $L \in \mathbb{R}$ so that for any given $\varepsilon > 0$, there is a $\delta > 0$ such that for all partitions P with $||P|| < \delta$,

$$|R(f,P,z) - L| < \varepsilon/2,$$

holds for any tags z. Let (P_1, z_1) be another tagged partition. By the triangle inequality we have

$$|R(f,P,z) - R(f,P_1,z_1)| \le |R(f,P,z) - L| + |R(f,P_1,z_1) - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since the tags are arbitrary, it implies

$$\overline{R}(f,P) - \underline{R}(f,P) \le \varepsilon.$$

As a result,

$$0 \le \overline{R}(f) - \underline{R}(f) \le \overline{R}(f,P) - \underline{R}(f,P) \le \varepsilon.$$

Note that the first inequality comes from the definition of the upper/lower Riemann integrals. Since $\varepsilon > 0$ is arbitrary, $\overline{R}(f) = \underline{R}(f)$.

Conversely, using $\overline{R}(f) = \underline{R}(f)$ in part a, we know that for $\varepsilon > 0$, there exists a δ such that

$$0 \le \overline{R}(f, P) - \underline{R}(f, P) < \varepsilon,$$

for all partitions P, $\|P\| < \delta$. We have

$$\begin{aligned} R(f, P, z) - \underline{R}(f) &\le \overline{R}(f, P) - \underline{R}(f) \\ &\le \overline{R}(f, P) - \underline{R}(f, P) \\ &< \varepsilon, \end{aligned}$$

and similarly,

$$\overline{R}(f) - R(f, P, z) \le \overline{R}(f, P) - \underline{R}(f, P) < \varepsilon.$$

As $\overline{R}(f) = \underline{R}(f)$, combining these two inequalities yields

$$|R(f, P, z) - \underline{R}(f)| < \varepsilon,$$

for all P, $\|P\| < \delta$, so f is integrable, where $L = \underline{R}(f)$.

(c) For any bounded f on $[a, b]$ and $x \in [a, b]$, its **oscillation** at x is defined by

$$\begin{aligned} \omega(f, x) &= \inf_{\delta} \{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\} \\ &= \lim_{\delta \to 0^+} \{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\}. \end{aligned}$$

It is clear that $\omega(f, x) = 0$ if and only if f is continuous at x. The set of discontinuity of f, D, can be written as $D = \bigcup_{k=1}^{\infty} O(k)$, where $O(k) = \{x \in [a, b] : \omega(f, x) \ge 1/k\}$. Suppose that f is Riemann integrable on $[a, b]$. It suffices to show that each $O(k)$ is of measure zero. Given $\varepsilon > 0$, by

Integrability of f, we can find a partition P such that

$$\overline{R}(f,P) - \underline{R}(f,P) < \varepsilon/2k.$$

Let J be the index set of those subintervals of P which contains some elements of $O(k)$ in their interiors. Then

$$\begin{aligned}
\frac{1}{k}\sum_{j\in J}|I_j| &\leq \sum_{j\in J}(\sup_{I_j} f - \inf_{I_j} f)\Delta x_j \\
&\leq \sum_{j=1}^{n}(\sup_{I_j} f - \inf_{I_j} f)\Delta x_j \\
&= \overline{R}(f,P) - \underline{R}(f,P) \\
&< \varepsilon/2k.
\end{aligned}$$

Therefore

$$\sum_{j\in J}|I_j| < \varepsilon/2.$$

Now, the only possibility that an element of $O(k)$ is not contained by one of these I_j is it being a partition point. Since there are finitely many partition points, say N, we can find some open intervals $I'_1, ..., I'_N$ containing these partition points which satisfy

$$\sum |I'_i| < \varepsilon/2.$$

So $\{I_j\}$ and $\{I'_i\}$ together form a covering of $O(k)$ and its total length is strictly less than ε. We conclude that $O(k)$ is of measure zero.

Conversely, given $\varepsilon > 0$, fix a large k such that $\frac{1}{k} < \varepsilon$. Now the set $O(k)$ is of measure zero, we can find a sequence of open intervals $\{I_j\}$ satisfying

$$O(k) \subseteq \bigcup_{j=1}^{\infty} I_j,$$

$$\sum_{j=1}^{\infty} |I_{i_j}| < \varepsilon.$$

One can show that $O(k)$ is closed and bounded, hence it is compact. As a result, we can find $I_{i_1}, ..., I_{i_N}$ from $\{I_j\}$ so that

$$O(k) \subseteq I_{i_1} \cup ... \cup I_{i_N},$$

$$\sum_{j=1}^{N} |I_j| < \varepsilon.$$

Without loss of generality we may assume that these open intervals are mutually disjoint since, whenever two intervals have nonempty intersection, we can put them together to form a larger open interval. Observe that $[a, b] \setminus (I_{i_1} \cup \cdots \cup I_{i_N})$ is a finite disjoint union of closed bounded intervals, call them $V_i's,\ i \in A$. We will show that for each $i \in A$, one can find a partition on each $V_i = [v_{i-1}, v_i]$ such that the oscillation of f on each subinterval in this partition is less than $1/k$.

Fix $i \in A$. For each $x \in V_i$, we have

$$\omega(f, x) < \frac{1}{k}.$$

By the definition of $\omega(f, x)$, one can find some $\delta_x > 0$ such that

$$\sup\{f(y) : y \in B(x, \delta_x) \cap [a, b]\} - \inf\{f(z) : z \in B(x, \delta_x) \cap [a, b]\} < \frac{1}{k},$$

where $B(y, \beta) = (y - \beta, y + \beta)$. Note that $V_i \subseteq \bigcup_{x \in V_i} B(x, \delta_x)$. Since V_i is closed and bounded, it is compact. Hence, there exist $x_{l_1}, \ldots, x_{l_M} \in V_i$ such that $V_i \subseteq \bigcup_{j=1}^{M} B(x_{i_j}, \delta_{x_{l_j}})$. By replacing the left end point of $B(x_{i_j}, \delta_{x_{l_j}})$ with v_{i-1} if $x_{l_j} - \delta_{x_{l_j}} < v_{i-1}$, and replacing the right end point of $B(x_{i_j}, \delta_{x_{l_j}})$ with v_i if $x_{l_j} + \delta_{x_{l_j}} > v_i$, one can list out the endpoints of $\{B(x_{l_j}, \delta_{l_j})\}_{j=1}^{M}$ and use them to form a partition S_i of V_i. It can be easily seen that each subinterval in S_i is covered by some $B(x_{l_j}, \delta_{x_{l_j}})$, which implies that the oscillation of f in each subinterval is less than $1/k$. So, S_i is the partition that we want.

The partitions S_i's and the endpoints of $I_{i_1}, ..., I_{i_N}$ form a partition P of $[a, b]$. We have

$$\begin{aligned}
\overline{R}(f, P) - \underline{R}(f, P) &= \sum_{I_{i_j}} (M_j - m_j)\Delta x_j + \sum (M_j - m_j)\Delta x_j \\
&\leq 2M \sum_{j=1}^{N} |I_{i_j}| + \frac{1}{k} \sum \Delta x_j \\
&\leq 2M\varepsilon + \varepsilon(b - a) \\
&= [2M + (b - a)]\varepsilon,
\end{aligned}$$

where $M = \sup_{[a,b]} |f|$ and the second summation is over all subintervals in $V_i, i \in A$. Hence f is integrable on $[a, b]$.