

Exercise 5

1. We summarize the properties of the Lebesgue measure on \mathbb{R}^n . For $E \subset \mathbb{R}^n$, let

$$\mathcal{L}^n(E) = \inf \left\{ \sum_k |C_k| : E \subset \bigcup_k C_k, C_k \text{ closed cubes} \right\} .$$

We have

- (a) $\mathcal{L}^n(C) = 1$ for every unit cube C (open or closed).
- (b) \mathcal{L}^n is a σ -finite Borel measure.
- (c) \mathcal{L}^n is finite on bounded sets.
- (d) For every measurable E ,

$$\mathcal{L}^n(E) = \inf \{ \mathcal{L}^n(G) : E \subset G, G \text{ open} \};$$

$$\mathcal{L}^n(E) = \sup \{ \mathcal{L}^n(K) : K \subset E, K \text{ compact} \} .$$

- (e) Let T be a linear transformation from \mathbb{R}^n to itself. For each measurable E , $T(E)$ is also measurable and there is some constant C_T such that

$$\mathcal{L}^n(T(E)) = C_T \mathcal{L}^n(E) .$$

(a)-(d) were covered in previous exercises. Prove (e).

2. Let Φ be a Lipschitz continuous map on \mathbb{R}^n to \mathbb{R}^n , that is, for some $L > 0$,

$$|\Phi(x) - \Phi(y)| \leq L|x - y| , \quad \forall x, y \in \mathbb{R}^n .$$

Show that $\Phi(E)$ is measurable if E is (Lebesgue) measurable.

3. This problem is related to the σ -finiteness condition in Proposition 2.10. Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \quad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2 .$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let x_1, \dots, x_n be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such $x!$), and define

$$\Lambda f = \sum_{j=1}^n \int_{-\infty}^{\infty} f(x_j, y) dy.$$

Let μ be the measure associated with this Λ by the representation theorem. If E is the x -axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

4. Let μ be a Borel measure on \mathbb{R}^n such that $\mu(K) < \infty$ for all compact K . Show that μ is the restriction of some Riesz measure on \mathcal{B} .