MATH 3310 Computational and Applied Mathematics

Final Exam Practice Exercises

Iterative method

2. Consider the linear system:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

   (a) Write down the Jacobi's iterative method and Gauss-Seidel iterative method to solve the linear system, in the form of $\mathbf{x}^{k+1} = B\mathbf{x}^k + \mathbf{c}$. Starting with $\mathbf{x}^0 = (0,0,0)^T$. Write down the first two iterations for each iterative scheme.

   (b) Determine whether the Jacobi's method and the Gauss-Seidel's method to solve the above linear system converge.

3. Consider the real matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & a \\ -1 & 1 & 0 & b \\ -2 & -1 & 1 & c \\ -3 & -2 & -1 & 1 \end{pmatrix}.$$

   where a, b and c are positive real constants.

   (a) Find a necessary and sufficient condition in term of a, b and c such that the Gauss-Seidel method for solving the linear system $A\mathbf{x} = \mathbf{b}$ converges. Explain your answer.

   (b) Prove that if $a = b = -c/11$, then the Gauss-Seidel method for solving $A\mathbf{x} = \mathbf{b}$ converges in finitely many iterations.

4. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where A is a SPD. Does the Jacobi method in solving the linear system always converge? If not, find a counter example.

5. Let $A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$. State the Householder-John theorem. Prove that the A is SPD. Using the Householder-John theorem. Show that both the Jacobi method and Gauss-Seidel method converge.

6. (a) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix}$$

   Write down the SOR iterative scheme to solve the linear system, in the form of $\mathbf{x}^{k+1} = B\mathbf{x}^k + \mathbf{c}$.

(b) Note that A is strictly diagonally dominant. In general, suppose A is a strictly diagonally dominant $n \times n$ matrix. Prove that the SOR algorithm converges for $0 < \omega \leq 1$. (Hint: Prove that all eigenvalues of B in the iterative scheme has magnitude straightly less than 1. You may use without proof the fact that: the iterative scheme converges if the spectral radius $\rho(B)$ of B satisfies $\rho(B) < 1$.)

7. Consider the linear system $A\mathbf{x} = \mathbf{b}$ ($\mathbf{x} \in \mathbb{R}^3$), where

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We solve the linear system using successive over-relaxation (SOR) method with an initial guess $\mathbf{x}^{(0)} = (0, 0, 0)^T$.

(a) Write down the numerical algorithm of the SOR method to solve the above linear system.

(b) Prove that the SOR method converges for solving the above linear system if and only if $0 < \omega < 2$ (ω is the parameter in the SOR method).

(c) Find the optimal choice of ω such that the SOR method converges at the fastest rate. (**Note** : You may use the theorems in the lecture notes without proof.)

8. Let A be an invertible $n \times n$ upper triangular matrix. Consider the linear system $A\mathbf{x} = \mathbf{b}$.

(a) Show that both the Jacobi's method and Gauss-Seidel's method to solve the linear system converge.

(b) Consider the SOR method with relaxation parameter ω. Find the optimal ω so that the SOR method has the fastest convergency rate.

(c) Now, consider the linear system $A^T\mathbf{x} = \mathbf{b}$. Determine whether the the Jacobi's method and Gauss-Seidel's method to solve the linear system converge.

9. Prove that $A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$ is positive definite when $-1/2 < a < 1$. Also, find a necessary and sufficient condition on a such that the Jacobi method converges.

10. Let A be $n \times n$ real matrix with real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ (may not be distinct). Assume that $0 < \alpha \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq \beta$. Consider the iterative scheme:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \omega(\mathbf{b} - A\mathbf{x}^k), \quad k = 0, 1, 2, ...$$

(a) Prove that the iterative scheme converges to the solution of $A\mathbf{x} = \mathbf{b}$ if $0 < \omega < 2/\beta$.

(b) Assume that $\lambda_1 = \lambda_2 = ... = \lambda_n$. Prove that the iterative scheme converges in n iterations if $\omega = 1/\lambda_1$.

Eigenvalues Problem

11. Let

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 2 & 4 & -1 \\ 2 & 5 & -2 \end{pmatrix}$$

(a) Starting with $\mathbf{x}^0 = (1, 1, 1)^T$. Compute all iterations of the power method to approximate the dominant eigenvector of A. Does the power method converge in this case? Explain your answer.

(b) By considering the power method on A^2 with $\mathbf{x}^0 = (1, 1, 1)^T$, determine the dominant eigenvalues of A and their corresponding eigenvectors. Explain your answer.

12. Find the QR factorization of

$$B = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 7 \\ 0 & -1 & -1 \end{pmatrix}$$

13. Define the Rayleign quotient $R(\mathbf{v}, A)$. Find a upper bound and lower bound for $R(\mathbf{v}, A)$ in term of the eigenvalues of A. Describe the RQI algorithm.

14. Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$. Describe the inverse power method. Compute the first two iterations of the inverse power method. (You may use calculator for this example. In the final exam, the numbers are nice and no calculator is needed.)

15. Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix}$. We use inverse power method with shift to find the eigenvalue closest to 1.3. Describe the inverse power method with shift algorithm. Compute the first two iterations of the inverse power method with shift. (You may use calculator for this example. In the final exam, the numbers are nice and no calculator is needed.)

16. Describe how to use power method to find the smallest eigenvalues of a non-singular matrix A. (Hint: Inverse power method with shift.)

17. Let

$$A = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}; \qquad B = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

   (a) Starting with $\mathbf{x}^0 = (1,1)^T$. Compute the first two iterations of the power method to approximate the dominant eigenvector of A.

   (b) Starting with $\mathbf{x}^0 = (1,1,1)^T$. Compute the first two iterations of the power method to approximate the dominant eigenvector of B.

   (c) Find the eigenvalues and eigenvectors of A and B. Determine whether the iterations in (a) and (b) converge to the dominant eigenvector of A and B respectively. Explain your answer.

18. Consider an invertible $n \times n$ symmetric matrix A with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, where:

$$|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n|$$

Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ be the eigenvectors associated with $\lambda_1, \lambda_2, ..., \lambda_n$ respectively, which form a basis of $\mathbb{R}^n$.

   (a) Show that if $|\lambda_1| = |\lambda_2|$, the power method with infinity norm does not converge in general. (Hint: Show that if λ_1 = -λ_2, the power method with infinity norm does not converges.)

   (b) Determine whether the power method with infinity norm converge if $\lambda_1 = \lambda_2$ ($\lambda_1 > 0$). Explain you answer.

19. Let $A \in M_{n\times n}(\mathbb{R})$ be real symmetric positive definite matrix with eigenvalues $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_n|$. Consider: $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$ $(k = 1, 2, ...)$. Prove that:

$$\frac{A\mathbf{x}^{(k)} \cdot \mathbf{x}^{(k)}}{\mathbf{x}^{(k)} \cdot \mathbf{x}^{(k)}} = \lambda_1 + \mathcal{O}(|\frac{\lambda_2}{\lambda_1}|^{2k})$$

20. Let

$$A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 3 & 4 & 1 \\ 1 & 1 & 2 & -3 \end{pmatrix}$$

   (a) Find the QR factorization of A.

   (b) Using (a), solve the equation $A\mathbf{x} = (1,2,3,4)^T$.

(c) Compute the first iteration of the QR method to approximate the eigenvalues of A.

21. Let:

$$A = \begin{pmatrix} 3 & -2 \\ 0 & 3 \\ 4 & 4 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

We consider the least square problem: find $\mathbf{x}$ that minimizes $E(\mathbf{x}) = ||A\mathbf{x} - \mathbf{b}||^2$.

(a) Find the QR factorization of A. Hence, solve the above least square problem.

(b) Let $B = \begin{pmatrix} 3 & -2 & 12 \\ 0 & 3 & 20 \\ 4 & 4 & -9 \end{pmatrix}$.

Given that $(12, 20, -9)^T = (-2, 3, 4)^T \times (3, 0, 4)^T$ (cross product). Find the QR factorization of B.

(c) Describe the QR method to approximate the eigenvalues of B. Using (b), compute the first iteration of the QR method to approximate the eigenvalues of B.

22. Let $\{A_k\}_{k=1}^{\infty}$ be the sequence obtained from the QR method on A. Show that

$$A_k = Q^T A Q$$

for some orthogonal matrix Q.

23. Does QR method always converge to a upper triangular matrix? Explain your answer.