THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3310 2022-2023
Homework Assignment 2 Suggested Solution

1. Solve the following PDE using Spectral Method:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = u(1,t), & t \in [0,\infty) \\ u(x,0) = f(x), & x \in [0,1] \end{cases}$$

where

$$f(x) = \begin{cases} -x(2x-1), & \text{if } x \in [0,\frac{1}{2}] \\ 0, & \text{else} \end{cases}$$

Solution:
Let $u(x,t) = X(x)T(t)$.
Putting in differential equation,

$$\frac{X''}{X} = \frac{T'}{T} = \lambda, \quad \lambda \text{ constant.}$$

Suppose $\lambda \leq 0$. Solving $X''(x) = \lambda X(x)$, and putting boundary conditions, we have

$$X(x) = \begin{cases} \alpha, & \text{if } \lambda = 0 \\ \alpha_1 \cos(2\pi n x) + \alpha_2 \sin(2\pi n x), \sqrt{-\lambda} = 2\pi n, & \text{if } \lambda < 0 \end{cases}$$

Then:

$$T(t) = \begin{cases} \beta, & \text{if } \lambda = 0 \\ \beta e^{-4\pi^2 n^2 t}, & \text{if } \lambda < 0 \end{cases}$$

Therefore:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(2\pi n x) + B_n \sin(2\pi n x)) e^{-4\pi^2 n^2 t}$$

Putting initial condition and comparing Fourier Coefficients with $f(x)$, we have:

$$\begin{aligned} A_0 &= \frac{1}{24} \\ A_n &= -\frac{1}{2(\pi n)^2}((-1)^n + 1) \\ B_n &= \frac{1}{(\pi n)^3}((1 - (-1)^n) \end{aligned}$$

2. Recall the definitions of discrete and inverse discrete Fourier Transform from the lecture notes: Given: $f_0, f_1, \ldots, f_{n-1} \in C$, the discrete Fourier transform is defined as

$$c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j e^{-i\frac{2jk\pi}{n}}$$

for $k = 0, 1, 2, \ldots, n-1$. And the inverse discrete Fourier Transform:

$$f_j = \sum_{k=0}^{n-1} c_k e^{i\frac{2jk\pi}{n}}$$

for $j = 0, 1, 2, \ldots, n-1$.
Check that the inverse discrete Fourier Transform does recover the discrete Fourier Transform.

Solution: By direct substitution, for fixed j

$$\begin{aligned}\sum_{k=0}^{n-1} c_k e^{i\frac{2jk\pi}{n}} &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{t=0}^{n-1} f_t e^{-i\frac{2tk\pi}{n}} e^{i\frac{2jk\pi}{n}} \\ &= \frac{1}{n}\sum_{t=0}^{n-1} f_t \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}(j-t)} \\ &= f_j\end{aligned}$$

3. Let $f = \{f_i\}_{i=0}^{n-1}$ and $g = \{g_i\}_{i=0}^{n-1}$ be two sequences of points in C that are periodic. Define convolution by

$$(f * g)_i = \sum_{k=0}^{n-1} f_k g_{i-k}$$

Prove that for $k = 0, \ldots, n-1$

$$\widehat{(f * g)}(k) = n\hat{f}(k)\hat{g}(k)$$

where $\hat{f} = \text{DFT}(f)$.

Solution:

$$\begin{aligned}\widehat{(f * g)}(k) &= \frac{1}{n}\sum_{j=0}^{n-1}\left(\sum_{d=0}^{n-1} f_d g_{j-d}\right)\exp\left(-i\frac{2jk\pi}{n}\right) \\ &= \frac{1}{n}\sum_{d=0}^{n-1} f_d \sum_{j=0}^{n-1} g_{j-d}\exp\left(-i\frac{2jk\pi}{n}\right) \\ &= \frac{1}{n}\sum_{d=0}^{n-1} f_d \sum_{j=0}^{n-1} g_{j-d}\exp\left(-i\frac{2dk\pi}{n}\right)\exp\left(-i\frac{2(j-d)k\pi}{n}\right) \\ &= \frac{1}{n}\sum_{d=0}^{n-1} f_d \exp\left(-i\frac{2dk\pi}{n}\right)\sum_{j=0}^{n-1} g_{j-d}\exp\left(-i\frac{2(j-d)k\pi}{n}\right) \\ &= n\cdot\frac{1}{n}\sum_{d=0}^{n-1} f_d \exp\left(-i\frac{2dk\pi}{n}\right)\cdot\frac{1}{n}\sum_{j=0}^{n-1} g_j \exp\left(-i\frac{2jk\pi}{n}\right) \\ &= n\hat{f}(k)\hat{g}(k)\end{aligned}$$

4. In addition to 1D DFT, we can also see an example that is 2D DFT. Consider this alternative definition for the DFT on $N \times N$ images:

$$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} f(k,l)e^{2\pi i\frac{mk+nl}{N}}$$

.

(a) Show that the inverse DFT (iDFT) is defined by

$$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \hat{f}(m,n)e^{-2\pi i\frac{pm+qn}{N}}.$$

(b) Determine the matrix U used to calculate the DFT of an $N \times N$ image, i.e. $\hat{f} = UfU$.

(c) Show that U is unitary (that is, $UU^* = U^*U = I$, where U^* is the conjugate transpose of U).

Solution:

(a)

$$\begin{aligned}
&\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\hat{f}(m,n)e^{-2\pi i\frac{pm+qn}{N}}\\
=&\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}f(k,l)e^{2\pi i\frac{mk+nl}{N}}\right)e^{-2\pi i\frac{pm+qn}{N}}\\
=&\frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}f(k,l)e^{2\pi i\frac{m(k-p)+n(l-q)}{N}}.
\end{aligned}$$

When $k=p$ and $l=q$, we have $f(k,l)e^{2\pi i\frac{m(k-p)+n(l-q)}{N}}=f(p,q)$ for any m,n.
When $k\neq p$,

$$\sum_{m=0}^{N-1}f(k,l)e^{2\pi i\frac{m(k-p)+n(l-q)}{N}}=e^{2\pi i\frac{n(l-q)}{N}}\sum_{m=0}^{N-1}f(k,l)e^{2\pi i\frac{m(k-p)}{N}}=0.$$

Similarly, when $l\neq p$,

$$\sum_{n=0}^{N-1}f(k,l)e^{2\pi i\frac{m(k-p)+n(l-q)}{N}}=e^{2\pi i\frac{m(k-p)}{N}}\sum_{m=0}^{N-1}f(k,l)e^{2\pi i\frac{n(l-q)}{N}}=0.$$

Therefore, we have

$$\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\hat{f}(m,n)e^{-2\pi i\frac{pm+qn}{N}}=\frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}f(p,q)=f(p,q)$$

(b) Let $u_{r,s}$ be the entry at (r+1)-th row and (s+1)-th column of the matrix U, here $0\leq r,s\leq N-1$. Then from $\hat{f}=UfU$ we can easily get

$$u_{r,s}=\frac{1}{\sqrt{N}}e^{2\pi i\cdot\frac{rs}{N}}$$

(c) Let $u^*_{m,n}$ be the entry at (m+1)-th row and (n+1)-th column of the matrix U^*, here $0\leq m,n\leq N-1$. Then

$$u^*_{m,n}=\overline{u_{n,m}}=\frac{1}{\sqrt{N}}e^{-2\pi j\cdot\frac{mn}{N}}$$

Then it is easy to verify that $UU^*=U^*U=I$

5. Consider the differential equation:

$$(**)\qquad a\frac{d^2u}{dx^2}+b\frac{du}{dx}=f(x)\text{ for }x\in(0,2\pi),$$

where $a,b>0$. Assume u and f are periodically extended to R. Divide the interval $[0,2\pi]$ into n equal portions and let $x_j=\frac{2\pi j}{n}$ for $j=0,1,2,...,n-1$.
Let $\mathbf{u}=(u(x_0),u(x_1),...,u(x_{n-1}))^T$ and $\mathbf{f}=(f(x_0),f(x_1),...,f(x_{n-1}))^T$.
Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n\times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j=\frac{u(x_{j+2})-u(x_{j-2})}{4h}\quad\text{and}\quad(\mathcal{D}_2\mathbf{u})_j=\frac{u(x_{j+4})-2u(x_j)+u(x_{j-4})}{16h^2}.$$

for $j=0,1,2,...,n-1$.

(a) Explain why the differential equation (**) can be discretized as:

$$(***)\quad a\mathcal{D}_2\mathbf{u}+b\mathcal{D}_1\mathbf{u}=\mathbf{f}.$$

In other words, explain why $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively.

(b) Prove that $\overrightarrow{e^{ikx}} := (e^{ikx_0}, e^{ikx_1}, ..., e^{ikx_{n-1}})^T$ is an eigenvector of both $\mathcal{D}_1$ and $\mathcal{D}_2$ for $k = 0, 1, 2, ..., n-1$. What are their corresponding eigenvalues? Please explain your answer with details.

(c) Show that $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ forms a basis for C^n.

(d) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in C$. If $\mathbf{u}$ satisfies (***), show that

$$(a\lambda_k^2 + b\lambda_k)\hat{u}_k = \hat{f}_k \text{ where } \lambda_k = i\frac{\sin(2kh)}{2h},$$

for $k = 0, 1, 2, ..., n-1$. Please explain your answer with details.

Solution:

(a) By Taylor's expansion, we get

$$u(x_{j+2}) = u(x_j) + 2hu'(x_j) + o(2h)$$
$$u(x_{j-2}) = u(x_j) - 2hu'(x_j) + o(2h)$$

so we deduce that

$$u'(x_j) = \frac{u(x_{j+2}) - u(x_{j-2})}{4h} + o(1)$$

Then we can say that when we choose n is sufficiently large (or h is sufficiently small), $\mathcal{D}_1\mathbf{u}$ can approximate $\mathbf{u}'$, or $\mathcal{D}_1$ can approximate $\frac{d}{dx}$.
Similarly,

$$u(x_{j+4}) = u(x_j) + 4hu'(x_j) + \frac{16h^2}{2}u''(x_j) + o(h^2)$$
$$u(x_{j-4}) = u(x_j) - 4hu'(x_j) + \frac{16h^2}{2}u''(x_j) + o(h^2)$$

so

$$u''(x_j) = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2} + o(1)$$

Then we can say that when we choose n is sufficiently large (or h is sufficiently small), $\mathcal{D}_2\mathbf{u}$ can approximate $\mathbf{u}''$, or $\mathcal{D}_2$ can approximate $\frac{d^2}{dx^2}$

(b) By the structure of $\mathcal{D}_1\mathbf{u}$, it can be verified that

$$(\mathcal{D}_1\overrightarrow{e^{ikx}})_j = \frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4h}$$

So it suffices to show that

$$\frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4he^{ikx_j}}$$

is independent of the index j, and this value is exactly the eigenvalue of $\mathcal{D}_1$ corresponding $\overrightarrow{e^{ikx}}$.

$$\begin{aligned}\frac{e^{ikx_{j+2}} - e^{ikx_{j-2}}}{4he^{ikx_j}} &= \frac{e^{ik\cdot(x_j+2h)} - e^{ik\cdot(x_j-2h)}}{4he^{ikx_j}} \\ &= \frac{e^{i\cdot 2kh} - e^{i\cdot(-2kh)}}{4h} \\ &= \frac{i\sin(2kh)}{2h}.\end{aligned}$$

So $\overrightarrow{e^{ikx}}$ is the eigenvector of $\mathcal{D}_1$ corresponding the eigenvalue $\frac{i\sin(2kh)}{2h}$ for $k = 0, 1, ..., n-1$.
Similarly,

$$\begin{aligned}\frac{e^{ikx_{j+4}} - 2e^{ikx_j} + e^{ikx_{j-4}}}{16h^2e^{ikx_j}} &= \frac{e^{ik\cdot(x_j+4h)} - 2e^{ik\cdot x_j} + e^{ik\cdot(x_j-4h)}}{16h^2e^{ikx_j}} \\ &= \frac{e^{i\cdot 4kh} - 2 + e^{i\cdot(-4kh)}}{16h^2} \\ &= \frac{\cos(4kh) - 1}{8h^2}.\end{aligned}$$

So $\overrightarrow{e^{ikx}}$ is the eigenvector of $\mathcal{D}_2$ corresponding the eigenvalue $(\frac{i\sin(2kh)}{2h})^2 = \frac{\cos(4kh)-1}{8h^2}$ for $k = 0, 1, ..., n-1$.

(c) Since $\overrightarrow{e^{ikx}}$ are the eigenvectors of $\mathcal{D}_1$ corresponding the distinct eigenvalues, we get that they are linearly independent. So the set contains n linearly independent vectors forms a basis.

(d) By (b) we get $\mathcal{D}_1\overrightarrow{e^{ikx}} = \lambda_k\overrightarrow{e^{ikx}}$, $\mathcal{D}_2\overrightarrow{e^{ikx}} = (\lambda_k)^2\overrightarrow{e^{ikx}}$
so

$$\begin{aligned}
a\mathcal{D}_2\mathbf{u} + \mathbf{b}\boldsymbol{\mathcal{D}_1}\mathbf{u} &= a\mathcal{D}_2(\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}) + b\mathcal{D}_1(\sum_{k=0}^{n-1}\hat{u}_k\overrightarrow{e^{ikx}}) \\
&= a\sum_{k=0}^{n-1}(\lambda_k)^2\hat{u}_k\overrightarrow{e^{ikx}} + b\sum_{k=0}^{n-1}\lambda_k\hat{u}_k\overrightarrow{e^{ikx}} \\
&= \sum_{k=0}^{n-1}(a(\lambda_k)^2 + b\lambda_k)\hat{u}_k\overrightarrow{e^{ikx}} \\
&= \mathbf{f} \\
&= \sum_{k=0}^{n-1}\hat{f}_k\overrightarrow{e^{ikx}}
\end{aligned}$$

Since $\{\overrightarrow{e^{ikx}}\}_{k=0}^{n-1}$ is a basis, comparing the coefficients leads to the result that we want to prove.