Lecture 17:

Another condition for the convergence (useful for analyzing SOR)

Let $A = N - P$ (N is invertible)

Iterative scheme : $N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{b}$

Theorem: (Household - John) Suppose A and $(N^* + N - A)$ are self-adjoint positive-definite matrices, then the iterative scheme converges.

Proof: Consider $M = N^{-1}P = N^{-1}(N-A) = I - N^{-1}A$.

Suffice to show that all eigenvalues of M satisfy $|\lambda| < 1$ (λ can be complex). Let λ be an eigenvalue associated to the eigenvector $\vec{x}$. Then:

$$M\vec{x} = \lambda\vec{x} \Rightarrow (I - N^{-1}A)\vec{x} = \lambda\vec{x} \Rightarrow (N-A)\vec{x} = \lambda N\vec{x}$$

$$\Rightarrow (1-\lambda)N\vec{x} = A\vec{x}$$

Note that $\lambda \neq 1$. Otherwise 0 is an eigenvalue of A. Contradiction to the fact that A is positive definite.

Multiply $\vec{x}^*$ on both sides:

$$(1-\lambda)\vec{x}^* N\vec{x} = \vec{x}^* A\vec{x} \Rightarrow \vec{x}^* N\vec{x} = \frac{1}{(1-\lambda)}\vec{x}^* A\vec{x} \quad \text{--- } (1)$$

Take conjugate transpose on both sides:

$$(1-\bar{\lambda})\,\vec{x}^*N^*\vec{x} = \vec{x}^*A^*\vec{x} = \vec{x}^*A\vec{x}$$

$$\Rightarrow \quad \vec{x}^*N^*\vec{x} = \frac{1}{(1-\bar{\lambda})}\,\vec{x}^*A\vec{x} \quad —— (2)$$

(1) + (2) − $\vec{x}^*A\vec{x}$ on both sides:

$$\vec{x}^*(N+N^*-A)\vec{x} = \left(\frac{1}{1-\lambda} + \frac{1}{1-\bar{\lambda}} - 1\right)\vec{x}^*A\vec{x}$$

$$= \frac{1-|\lambda|^2}{|1-\lambda|^2}\,\vec{x}^*A\vec{x}$$

By assumption, A and $N+N^*-A$ are both positive definite.

We have: $\vec{x}^*A\vec{x} > 0$ and $\vec{x}^*(N+N^*-A)\vec{x} > 0$

Hence, $1-|\lambda|^2 > 0$ and $|\lambda| < 1$.

$\therefore$ $\rho(M) < 1$ and the iterative scheme converges.

Example: Let $A = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \alpha_{n-1} & \beta_{n-1} \\ 0 & & & \beta_{n-1} & \alpha_n \end{pmatrix}$ be real symmetric tridiagonal matrix. Suppose A is positive-definite. Prove that the Gauss-Seidel method converges.

Solution: For Gauss-Seidel method,

$$N = \begin{pmatrix} \alpha_1 & & & 0 \\ \beta_1 & \alpha_2 & & \\ & \ddots & \ddots & \\ 0 & & \beta_{n-1} & \alpha_n \end{pmatrix} \quad \text{and} \quad N^* + N - A = \begin{pmatrix} \alpha_1 & & & & 0 \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_{n-1} & \\ 0 & & & & \alpha_n \end{pmatrix}$$

Then, $N^* + N - A$ is symmetric.

Also, $\underbrace{(0, \cdots, \underset{\substack{\uparrow \\ i\text{th}}}{1}, 0 \cdots 0)}_{\vec{e}_i^{\,T}} A \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_i} \leftarrow i\text{th} = \alpha_i > 0$. $\therefore$ $N + N^* - A$ is positive definite

(∵ all eigenvalues are positive)

By Householder - John Theorem, Gauss-Seidel method converges.

Example: Suppose A is (real) self-adjoint positive-definite. Using Householder - John theorem, prove that: SOR method converges if and only if $0 < \omega < 2$.

Solution: Note that $N_{SOR} = L + \frac{1}{\omega} D$. Now, $L = U^*$ (A is self-adjoint)

$$\therefore \quad N_{SOR} + N_{SOR}^* - A = \left(\frac{2}{\omega} - 1\right) D$$

$\therefore$ $N_{SOR} + N_{SOR}^* - A$ is also self-adjoint positive-definite if $0 < \omega < 2$.

$\therefore$ By Householder - John theorem, SOR converges.

Eigenvalue Problem

Recall: Convergence of iterative scheme:

$N\vec{x}^{k+1} = P\vec{x}^{k} + \vec{f}$ depends on the spectral radius $\rho(N^{-1}P)$.

$\therefore$ Need: numerical method to compute eigenvalues.

Computation of Spectral radius

Goal: Find eigenvalues with largest magnitude $\leftarrow$ Spectral radius

Two methods:
1. Power method
2. QR method

1. Power method

Let $A \in M_{n\times n}(\mathbb{C})$ with n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$.

Let $X = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}(\mathbb{C})$. Then, we know:

$$AX = X\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Assuming: $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

We'll use Power method to compute $|\lambda_1|$.

<u>Observation:</u> Start with an initial vector $\vec{x}^{(0)}$.

Consider the iterative scheme: $\vec{x}^{(k+1)} = \dfrac{A\vec{x}^{(k)}}{\|A\vec{x}^{(k)}\|_\infty}$ for $k=0,1,\ldots$

Suppose A is diagonalizable. That's, we can assume $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ form a basis for $\mathbb{C}^n$.

Take $\vec{x}^{(0)} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n$ (assuming $a_1 \neq 0$)

Note that $A^k \vec{x}^{(0)} = a_1\lambda_1^k \left[\vec{x}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k \vec{x}_j \right]$.

Hence, $\vec{x}^{(k)} = \dfrac{A\vec{x}^{(k-1)}}{\|A\vec{x}^{(k-1)}\|_\infty} = \dfrac{A\left(A\vec{x}^{(k-2)} / \|A\vec{x}^{(k-2)}\|_\infty \right)}{\|A(A\vec{x}^{(k-2)}/\|A\vec{x}^{(k-2)}\|_\infty)\|_\infty}$

$= \dfrac{A^2\vec{x}^{(k-2)}}{\|A^2\vec{x}^{(k-2)}\|_\infty} = \cdots = \dfrac{A^k\vec{x}^{(0)}}{\|A^k\vec{x}^{(0)}\|_\infty}$

$$\vec{x}^{(k)} = \frac{a_1 \lambda_1^k \left[\vec{x}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k \vec{x}_j \right]}{\left\| a_1 \lambda_1^k \left[\vec{x}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k \vec{x}_j \right] \right\|_\infty} \approx \underbrace{\frac{a_1}{|a_1|} \frac{\vec{x}_1}{\|\vec{x}_1\|_\infty} \frac{\lambda_1^k}{|\lambda_1|^k}}_{\vec{v}}$$

when k is big

$\left(\because \left| \frac{\lambda_j}{\lambda_1} \right| < 1 \text{ for } j = 2, \ldots, n \right)$

Note: $\vec{v}$ is an eigenvector associated to λ_1.

In fact,

$$\| A \vec{x}^{(k)} \|_\infty \longrightarrow \left\| A \left(\frac{a_1}{|a_1|} \frac{\vec{x}_1}{\|\vec{x}_1\|_\infty} \right) \right\|_\infty = \left\| \frac{a_1}{|a_1|} \lambda_1 \frac{\vec{x}_1}{\|\vec{x}_1\|_\infty} \right\|_\infty = |\lambda_1|$$

$\frac{\lambda_1^k}{|\lambda_1|^k}$ can be removed under $\| \cdot \|_\infty$

Example: Consider: $A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & 10 \end{pmatrix}$. Using power method with $\vec{x}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, find the spectral radius of A.

Solution: $\vec{x}^{(1)} = \dfrac{A\vec{x}^{(0)}}{\|A\vec{x}^{(0)}\|_\infty} = \dfrac{(6, 8, 12)^T}{\|(6, 8, 12)^T\|_\infty} = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \\ 1 \end{pmatrix}$.

$\vec{x}^{(2)} = \dfrac{A\vec{x}^{(1)}}{\|A\vec{x}^{(1)}\|_\infty} = \dfrac{(7/3, 10/3, 16/3)^T}{\|(7/3, 10/3, 16/3)^T\|_\infty} = \begin{pmatrix} 7/16 \\ 5/8 \\ 1 \end{pmatrix}$.

Compute $\|A\vec{x}^{(k)}\|_\infty$, we have:

k	6	8	10
$\|A\vec{x}^{(k)}\|_\infty$	4.025316	4.006270	4.001564

. The dominant eigenvalue ≈ 4