Pf of Thm 1.16

(satisfying Lip condition)

Step 1: $\forall \varepsilon > 0$, $\exists$ a 2π-periodic Lip cts function g s.t.

$$\|f - g\|_2 < \varepsilon/2$$

Pf: By Lemma 1.3 (and its proof), $\forall \varepsilon_1 > 0$

$\exists$ step function

$$s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j}(x)$$

where
$$\begin{cases} m_j = \inf\{ f(x) : x \in [a_j, a_{j+1}] \} \\ I_j = (a_j, a_{j+1}] \text{ for } j = 1, \dots, N-1 \\ I_0 = [a_0, a_1] . \end{cases}$$

such that
$$\begin{cases} s \le f \text{ and} \\ \int_{-\pi}^{\pi} f - s < \varepsilon_1 \end{cases}$$

Since f is Riemann integrable, f is bounded.

i.e. $\exists M > 0$ s.t. $-M \le f \le M$.

This implies $\quad -M \le m_j \le M$

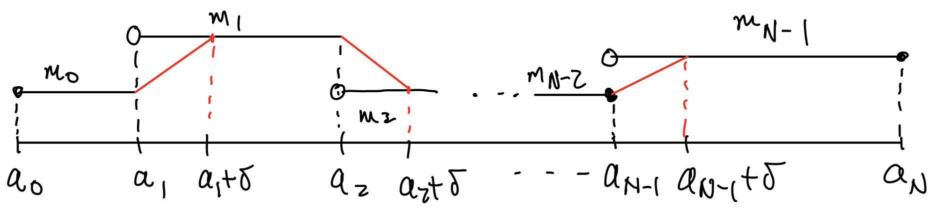
and hence $\quad -M \le s \le M$.

Note that $f \ge s$, we then have $0 \le f - s \le M$.

$$\Rightarrow \int_{-\pi}^{\pi} (f-S)^2 \le M \int_{-\pi}^{\pi} f-S < M\varepsilon,$$

Then choose $\delta > 0$ such $\delta < a_{j+1} - a_j$, $j = 1, 2, \cdots, N-1$ and define a piecewise linear continuous function by

$$g(x) = \begin{cases} \dfrac{m_j - m_{j-1}}{\delta}(x - a_j) + m_{j-1}, & \text{for } x \in (a_j, a_j + \delta),\ j = 1, \cdots, N-1 \\ S(x), & \text{otherwise} \end{cases}$$



Then clearly, g satisfies a Lipschitz condition.

And

$$\int_{-\pi}^{\pi} (S-g)^2 = \sum_{j=1}^{N-1} \int_{a_j}^{a_j+\delta} \left(S(x) - \frac{m_j - m_{j-1}}{\delta}(x - a_j) - m_{j-1} \right)^2$$

$$= \sum_{j=1}^{N-1} \int_{a_j}^{a_j+\delta} \left(m_j - \frac{m_j - m_{j-1}}{\delta}(x - a_j) - m_{j-1} \right)^2$$

$$= \sum_{j=1}^{N-1} (m_j - m_{j-1})^2 \int_{a_j}^{a_j+\delta} \left(1 - \frac{x - a_j}{\delta} \right)^2$$

$$= \sum_{j=1}^{N-1} (m_j - m_{j-1})^2 \int_{a_j}^{a_j+\delta} \left(\frac{\delta + a_j - x}{\delta} \right)^2$$

$$\leq \delta \sum_{j=1}^{N-1} (m_j - m_{j-1})^2$$

$$\leq M^2(N-1)\delta$$

Therefore
$$\int_{-\pi}^{\pi} (f-g)^2 = \int_{-\pi}^{\pi} ((f-s)+(s-g))^2$$
$$\leq 2\int_{-\pi}^{\pi} (f-s)^2 + (s-g)^2$$
$$< 2M\varepsilon_1 + 2M^2(N-1)\delta$$

Now, for any $\varepsilon > 0$,

we first choose $\varepsilon_1 = \dfrac{\varepsilon^2}{4M}$

Then find the step as described with N & a_j accordingly.

Finally, choose

$$\delta = \min\left\{ \frac{\varepsilon^2}{4M^2(N-1)},\ a_{j+1} - a_j \right\}_{j=1,\cdots,N}$$

we conclude, the Lip. function g satisfies

$$\int_{-\pi}^{\pi} (f-g)^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$$

$$\Rightarrow \quad \|f-g\|_2 < \varepsilon$$

$\#$

(In fact, our proof shows that if $s(x)$ is a step function on $[a,b]$, then $\forall \varepsilon > 0$, $\exists$ Lip function $g(x)$ s.t. $\|s-g\|_\infty < \varepsilon/2$)

Step 2 Completion of the proof.

Applying Thm 1.7 to the function g in Step 1:

$\exists N > 0$ s.t. $\|g - S_N g\|_\infty < \frac{\varepsilon}{2\sqrt{2\pi}}$

(Not the N in Step 1)

Thus $\|g - S_N g\|_2 = \left[\int_{-\pi}^{\pi} (g - S_N g)^2 \right]^{1/2} \leq \left[2\pi \|g - S_N g\|_\infty^2 \right]^{1/2}$

$< \frac{\varepsilon}{2}$

By Cor 1.15,

(Note: g may not belongs to E_N, but $S_N g \in E_N$)

$\|f - S_N f\|_2 \leq \|f - S_N g\|_2$

$\leq \|f - g\|_2 + \|g - S_N g\|_2$ (Ex!)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ (Step 1).

Finally, $\forall n \geq N$, set of generators of $E_N \subset$ set of generators of E_n,

$\therefore$ $E_N \subset E_n$.

Hence $\forall n \geq N$, $\|f - S_n f\|_2 \leq \|f - S_N f\|_2 < \varepsilon$

$\in E_N \subset E_n$

proj of f onto E_n

i.e. $\lim_{n \to \infty} \|S_n f - f\|_2 = 0$ ※

Cor 1.17 (a) Suppose that f_1 & f_2 are 2π-periodic integrable functions on $[-\pi, \pi]$ with the <u>same Fourier Series</u>. Then

<u>$f_1 = f_2$ almost everywhere.</u>

(i.e. $f_1 = f_2$ except a set of measure zero.)

(b) Suppose that f_1 & f_2 are 2π-periodic <u>continuous</u> functions with the <u>same Fourier series</u>. Then <u>$f_1 = f_2$</u>

<u>Recall</u>: A set E is said to be of <u>measure zero</u> if $\forall \varepsilon > 0$, $\exists$ <u>countably</u> many intervals $\{I_k\}$ s.t.

$$E \subset \bigcup_k I_k \quad \& \quad \sum_k |I_k| < \varepsilon.$$

<u>Pf</u>: (a) Let $f = f_1 - f_2$, then $a_n(f) = b_n(f) = 0 \quad \forall n \geq 0$

$$\Rightarrow \quad S_n f = 0, \ \forall n \geq 0$$

Hence (Thm 1.16) $\lim_{n\to\infty} \|S_n f - f\|_2 = 0 \Rightarrow \|f\|_2 = 0$

By theory of Riemann integral, $f = 0$ almost everywhere.

(b) We still have $\|f\|_2 = 0$. As f_1, f_2 cts $\Rightarrow f^2$ cts, ≥ 0

$$\Rightarrow f^2 \equiv 0. \quad \#$$

$\boxed{\begin{array}{l}\underline{\text{Cor 1.18}}\ (\underline{\text{Parseval's Identity}}) \\ \text{For every } 2\pi\text{-periodic function } f \text{ integrable on } [-\pi, \pi] \\ \qquad \boxed{\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)} \\ \text{where } a_0, a_n, b_n \text{ are Fourier coefficients of } f.\end{array}}$

$\underline{\text{Pf}}$: By def of a_n, b_n

$$\begin{cases} \sqrt{2\pi}\, a_0 = \langle f, \frac{1}{\sqrt{2\pi}} \rangle_2 \\ \sqrt{\pi}\, a_n = \langle f, \frac{1}{\sqrt{\pi}} \cos nx \rangle_2 & n \geq 1 \\ \sqrt{\pi}\, b_n = \langle f, \frac{1}{\sqrt{\pi}} \sin nx \rangle_2 \end{cases}$$

Then $\langle f, S_N f \rangle_2 = \langle (f - S_N f) + S_N f, S_N f \rangle_2$

By Cor1.15, $S_N f = P_N f$ on E_N,

$\therefore$ $f - S_N f$ orthogonal to the subspace E_N (Ex!)

i.e. $\langle f - S_N f, S_N f \rangle_2 = 0$

Hence $\langle f, S_N f \rangle_2 = \langle S_N f, S_N f \rangle_2 = \|S_N f\|_2^2$

$$= \int_{-\pi}^{\pi} \left(a_0 + \sum_{k=1}^{N} a_k \cos kx + b_k \sin kx \right)^2 dx$$

$$= 2\pi a_0^2 + \pi \sum_{k=1}^{N} (a_k^2 + b_k^2)$$

Then $0 \overset{\text{Thm 1.16}}{=\!=} \lim_{N\to\infty} \|f - S_N f\|_2^2$

$$= \lim_{N\to\infty} \left(\|f\|_2^2 - 2\langle f, S_N f\rangle_2 + \|S_N f\|_2^2 \right)$$

$$= \lim_{N\to\infty} \left(\|f\|_2^2 - \|S_N f\|_2^2 \right)$$

$$\therefore \quad \|f\|_2^2 = \lim_{N\to\infty} \|S_N f\|_2^2 = \lim_{N\to\infty} \left[2\pi a_0^2 + \pi \sum_{k=1}^{N} (a_k^2 + b_k^2) \right]$$

#

eg: Fourier series of $f_1(x) = x$ on $[-\pi, \pi]$

$$f_1(x) = x \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx \qquad (a_n = 0 \ \forall n = 0, 1, \cdots)$$

Hence Parseval's Identity

$$\Rightarrow \quad \int_{-\pi}^{\pi} x^2 dx = \pi \sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{2}{n} \right]^2$$

$$\overset{(\text{Ex!})}{\Longrightarrow} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad (\text{Euler Famula})$$

Ch2 Metric Space

In this chapter, X always denotes a non-empty set.

Def: A metric on X is a function

$$d : X \times X \to [0, +\infty) \text{ such that}$$

$\forall\, x, y, z \in X$

(M1) $d(x,y) \geq 0$ & "equality holds $\Leftrightarrow x = y$".

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,y) \leq d(x,z) + d(z,y)$

The pair (X, d) is called a metric space.

Note: Condition (M3) is called the triangle inequality.

e.g. 2.1 $(X = \mathbb{R},\ d(x,y) = |x-y|)$ is a metric space.

e.g. 2.2 Let $X = \mathbb{R}^n$, $\quad d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ (Euclidean metric)

for $x = (x_1, \cdots, x_n)$ & $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

Then $(\mathbb{R}^n, d_2)$ is a metric space. (Proof omitted, Ex!)

eg. 2.3 Let $X = \mathbb{R}^n$, defines

$$\begin{cases} d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \\ d_\infty(x,y) = \max_{i=1,\cdots,n} |x_i - y_i| \end{cases}$$

Then $(\mathbb{R}^n, d_1)$ & $(\mathbb{R}^n, d_\infty)$ are metric spaces.

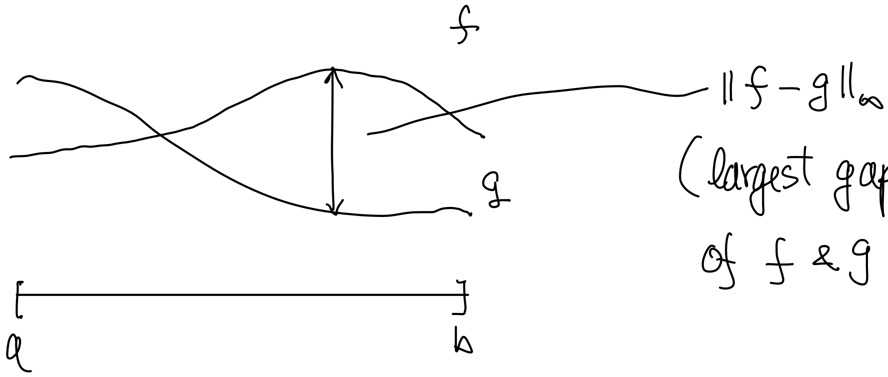
Generalization of egs 2.2 & 2.3 to function spaces:

eg 2.4 Let $C[a,b] = \{$ (real) continuous functions on $[a,b] \}$

$\forall\ f, g \in C[a,b]$, define

$$d_\infty(f,g) = \|f-g\|_\infty = \max\{ |f(x)-g(x)| : x \in [a,b] \}$$

Then $(C[a,b], d_\infty)$ is a metric space (Ex!)



(largest gaps between graphs of f & g)

Similarly, one can define

$$d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx$$

It is also easy to verify that $(C[a,b], d_1)$ is a metric space. (Ex!)

The natural generalization of the Euclidean metric to $C[a,b]$ is

$$d_2(f,g) = \sqrt{\int_a^b |f-g|^2}$$

Note that $d_2(f,g) = \|f-g\|_2$ (as in Fourier series)

(M1) & (M2) are clear for d_2 (because f, g cts.).

An Cauchy-Schwarz $\Rightarrow$ $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$ (Ex!)

$\Rightarrow$ d_2 satisfies (M3)

$\therefore$ $(C[a,b], d_2)$ is a metric space.

<u>Note</u>: We are restricted to the space $C[a,b]$ of continuous functions, <u>not</u> the bigger space $R[a,b]$ of Riemann integrable functions.

eg 2.5 On $X = R[a,b] = \{$ Riemann integrable functions on $[a,b] \}$.

d_1 is still defined

$$d_1(f,g) = \int_a^b |f-g|$$

However, (M1) is not satisfied:

$$d_1(f,g) = 0 \Leftrightarrow f = g \text{ almost everywhere}$$

$$\not\Rightarrow f = g \text{ (at every point)}$$

$\therefore$ d_1 is not a metric on $R[a,b]$.

To overcome this, we consider $X = R[a,b]/\sim$

where "$\sim$" is an equivalent relation on $R[a,b]$

defined by $f \sim g \Leftrightarrow f = g$ almost everywhere.

(Check: "$\sim$" is an equivalent relation.)

Then elements of $R[a,b]/\sim$ can be represented as ($f \in R[a,b]$)

$$[f] \text{ or } \bar{f} = \{ g \in R[a,b] : g = f \text{ almost everywhere} \}$$

Now define $\hat{d_1}$ on $R[a,b]/\sim$ by $\hat{d_1}(\bar{f}, \bar{g}) = d_1(f,g)$

Check: $\tilde{d}_1$ is well-defined

i.e. indep. of the choice of representatives f & g:

$\forall\ f_1 \in \bar{f}\ ,\ g_1 \in \bar{g}$.

$$
\begin{aligned}
d_1(f_1, g_1) &= \int_a^b |f_1 - g_1| \\
&\leq \cancelto{0}{\int_a^b |f_1 - f|} + \int_a^b |f - g| + \cancelto{0}{\int_a^b |g - g_1|} \\
&= d_1(f, g)
\end{aligned}
$$

Similarly $\quad d_1(f, g) \leq d_1(f_1, g_1)$

$\therefore \quad d_1(f, g) = d_1(f_1, g_1)$.

Then it is straight forward to verify that $\left(R[a,b]/\sim\ ,\ \tilde{d}_1\right)$ is a metric space.

Similarly for $\left(R[a,b]/\sim\ ,\ \tilde{d}_2\right)$ is a metric space & note that $\tilde{d}_2$ is the L^2- distance defined before:

$$
\tilde{d}_2(\bar{f}, \bar{g}) = \left(\int_a^b |f - g|^2 \right)^{1/2}
$$

Def: A <u>norm</u> $\|\cdot\|$ is a function on a <u>real vector space</u> X to $[0,\infty)$ s.t. $\forall\, x,y \in X$ & $\alpha \in \mathbb{R}$

(N1) $\|x\| \geq 0$ & "$\|x\| = 0 \Leftrightarrow x = 0$".

(N2) $\|\alpha x\| = |\alpha|\, \|x\|$

(N3) $\|x+y\| \leq \|x\| + \|y\|$

The pair $(X, \|\cdot\|)$ is called a <u>normed space</u>.

And $d(x,y) \overset{\text{def}}{=} \|x-y\|$ is called the <u>metric induced</u> by the norm $\|\cdot\|$.

(Ex: Show that $d(x,y) = \|x-y\|$ is a metric with the property

$$d(\alpha x, \alpha y) = |\alpha|\, d(x,y), \quad \forall\, \alpha \in \mathbb{R} \text{)}$$

<u>egs</u>: (a)

$$\left.\begin{aligned} \|x\|_2 &= \left(\sum x_i^2\right)^{1/2} \\ \|x\|_1 &= \sum |x_i| \\ \|x\|_\infty &= \max\{|x_1|, \cdots, |x_n|\} \end{aligned}\right\} \text{ are norms on } \mathbb{R}^n$$

(b)

$$\left.\begin{aligned} \|f\|_2 &= \left(\int_a^b |f|^2\right)^{1/2} \\ \|f\|_1 &= \int_a^b |f| \\ \|f\|_\infty &= \sup\{|f(x)| : x \in [a,b]\} \end{aligned}\right\} \text{ are norms on } C[a,b]$$

Note: We've seen "norm" induces "metric" already. However, a "metric" may not induced from a "norm".

eg. X = non-empty set

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

discrete metric on X (Ex: check this is a metric)

- X not necessary a vector space, so d is not induced by norm.

- Even X is a vector space:

$$\left.\begin{matrix} 1 \\ 0 \end{matrix}\right\} = d(\alpha x, \alpha y) = |\alpha| \, d(x,y) = \begin{cases} |\alpha| \\ 0 \end{cases}$$

Contradiction for $|\alpha| \neq 1$ (for $x \neq y$)

Def: Let (X, d) be a metric space. Then for any <u>non-empty</u> $Y \subset X$, $(Y, d|_{Y\times Y})$ is called a <u>metric subspace</u> of (X, d).

<u>Notes</u>: (i) metric subspace is a metric space.

(ii) We simple write (Y, d) for $(Y, d|_{Y\times Y})$

(iii) A metric subspace of a normed space needs <u>not</u> be a normed space, unless it is a vector subspace.

eg: $(\mathbb{R}^3, d_2)$ is a normed space (3-dim. Euclidean sp.)

$S^2 \subset \mathbb{R}^3$ with induced metric is clearly not a normed space.