MATH4210: Financial Mathematics Tutorial 6

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Binomial Tree for Exotic Options

Question

In a two step binomial tree model with one step interest $r = 0.05, S_0 = 100, u = 1.1, d = 0.9$, consider an contingent claim that expires after two years and payoff is the value of the squared stock price $(S(T))^2$, if the stock price $S(T)$ is strictly higher than 100 when the option is exercised; otherwise, the option pays 0.

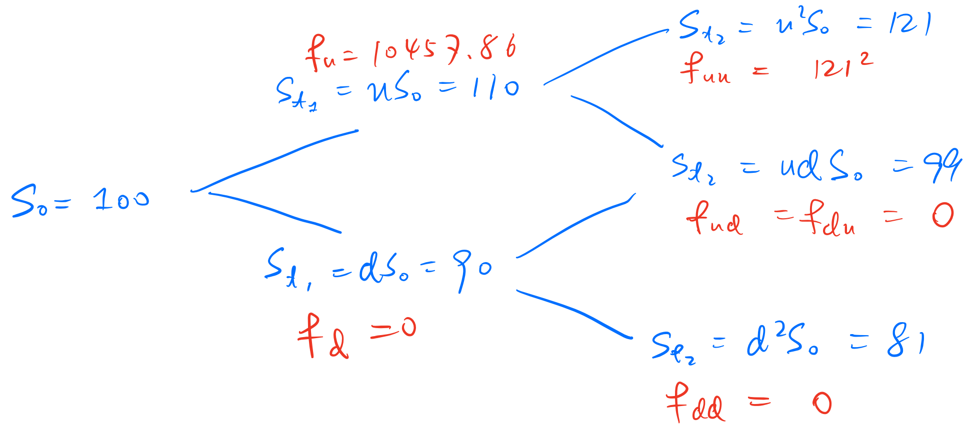
(1) Find the initial price and the replication strategy of the European version of the above option.

(2) In the same market model above, find the initial price and the replication strategy of the American version of the above option.

(1) The payoff function: $g(S_T) = S_T^2 \mathbb{1}_{S_T > 100}$

$$q = \frac{1 + r\Delta t - d}{u - d} = \frac{3}{4}.$$

Blue for stock price red for option price



$$f_u = (1+r\Delta t)^{-1}(121^2 \times q) = 10457.86$$

$$f = (1+r\Delta t)^{-1}(q \cdot f_u + (1-q)\cdot f_d) = 7469.9.$$

Replicating strategy:

$$\phi_0 = \frac{f_u - f_d}{uS_0 - dS_0} = 522.89, \quad \phi_{t_1}^u = \frac{f_{uu} - f_{ud}}{u^2 S_0 - udS_0} = 665.5, \quad \phi_{t_1}^d = \frac{f_{ud} - f_{dd}}{udS_0 - d^2 S_0} = 0$$

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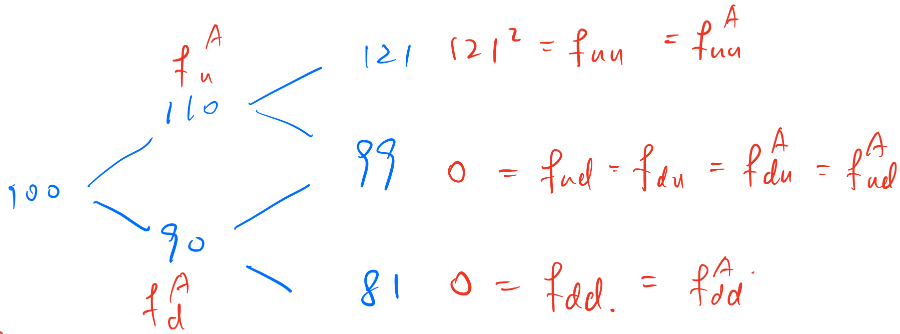
(2). Blue for stock, red for option price.



$$f_u^A = \max\{(110)^2, f_u\} = \max\{12100, 10457.86\} = 12100$$

$$f_d^A = 0, \quad f = (1+r\Delta t)^{-1}(q f_u^A + (1-q) f_d^A) = 8642.86.$$

Replicating strategy

$$\phi_{t_0} = \frac{f_u^A - f_d^A}{uS_0 - dS_0} = 605$$

$\phi_{t_1}^u = 0$ since we exercise immediately

$$\phi_{t_1}^d = \frac{f_{du}^A - f_{dd}^A}{udS_0 - u^2 S_0} = 0$$

Binomial Tree for Exotic Options

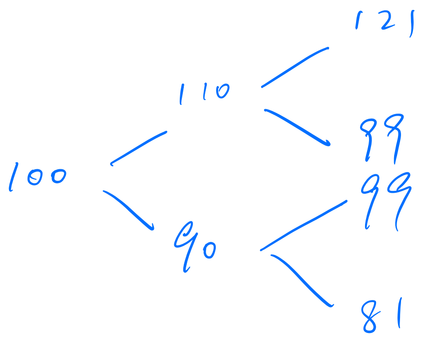
Question

A Lookback call is identical to a standard European call, except that the strike price is not set in advance, but is equal to the minimum price experienced by the underlying asset during the life of the call. Suppose the stock price $S_0 = 100, u = 1.1, d = 0.9$ *in each of the next two years, and one step interest* $r = 0.05$. *What is the price and the replication strategy of a two-year Lookback call option ?*

$K = \min_{i=0,1,2} \{S_{t_i}\}$.

payoff function: $g(S_T) = (S_T - \min_{i=0,1,2} \{S_{t_i}\})$.

Blue for Stock   red for option



$K = 100$ $f_{uu} = 21$

$K = 99$ $f_{ud} = 0$

$K = 90$ $f_{du} = 9$

$K = 81$ $f_{dd} = 0$

$$f_u = (1+r\delta t)^{-1}(q f_{uu} + (1-q) f_{ud}) = 15$$

$$f_d = (1+r\delta t)(q f_{du} + (1-q) f_{dd}) = 6.43$$

$$\Rightarrow f = (1+r\delta t)^{-1}(q f_u + (1-q) f_d) = 12.25$$

Replicating strategy:

$$\phi_{t_0} = \frac{f_u - f_d}{uS_0 - dS_0} = 0.4285$$

$$\phi_{t_1}^{u} = \frac{f_{uu} - f_{ud}}{u^2S_0 - udS_0} = 0.9545$$

$$\phi_{t_1}^{d} = \frac{f_{du} - f_{dd}}{duS_0 - d^2S_0} = 0.5$$

Binomial Tree for Exotic Options

Question

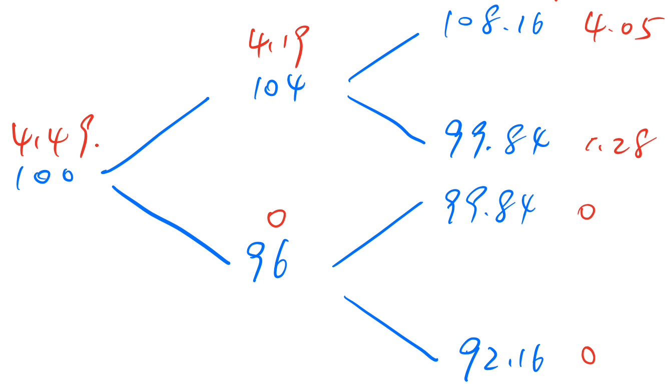
Suppose you are given a two step binomial tree model with the following: $S_0 = 100, u = 1.04, d = 0.96, r = 0.05$. *Consider a two period Asian call option where the averaging is done over all three prices observed, i.e., the initial price, the price after one period, and the price after two periods.*
(1) Suppose the option is an average-price Asian option with a strike of 100. Find the initial price and the replication strategy.
(2) Suppose the option is an average-strike Asian option. Find the initial price and the replication strategy.

$1 + r\Delta t = 1.05$, $u = 1.04$. $d = 0.96$.

$$\Rightarrow \frac{(1+r\Delta t) - d}{u - d} = 1.125 > 1$$

So q does not define a probability measure

We can still price the option by replicating.



(1). $A_{uu} = \frac{1}{3}(S_0 + uS_0 + u^2 S_0) = 104.05$

$A_{ud} = \frac{1}{3}(S_0 + uS_0 + udS_0) = 101.28$

$A_{du} = \frac{1}{3}(S_0 + dS_0 + duS_0) < K$

$A_{dd} = \frac{1}{3}(S_0 + dS_0 + d^2 S_0) < K.$

payoff:

$g((S_{t_i})_{i=0,1,2}) = (A_T - K)^+$

$$\Rightarrow \begin{cases} f_{uu} = (A_{uu} - K)^+ = 4.05 \\ f_{ud} = (A_{ud} - K)^+ = 1.28 \\ f_{du} = (A_{du} - K)^+ = 0 \\ f_{dd} = (A_{dd} - K)^+ = 0 \end{cases}$$

Assume we are at uS_0 at t_1, then we construct a replicating strategy:

$$\begin{cases} x = f_u \\ (x - \phi^u_{t_1} \cdot uS_0)(1 + r\Delta t) + \phi^u_{t_1} \cdot u^2 S_0 = f_{uu} \\ (x - \phi^u_{t_1} uS_0)(1 + r\Delta t) + \phi^u_{t_1} udS_0 = f_{ud} \end{cases}$$

Solve for the linear system, we still have

$f_u = (1 + r\Delta t)^{-1}(q f_{uu} + (1-q) f_{ud}) \neq E^Q[(1+r\Delta t)^{-1} f_{t_2} \mid S_{t_1} = uS_0]$

since Q is not well-defined.

$\Rightarrow f_u = 4.19$

and $\phi^u_{t_1} = \dfrac{f_{uu} - f_{ud}}{u^2 S_0 - udS_0} = 0.3329$

Similarly, we have $\phi^d_{t_1} = \dfrac{f_{du} - f_{dd}}{duS_0 - d^2 S_0} = 0$, $f_d = 0$

And $\phi_{t_0} = \frac{f_u - f_d}{uS_0 - dS_0} = 0.5238$, $f = 4.49$.

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(2) We sketch the solution:

$K_T = \frac{1}{3}(S_{t_0} + S_{t_1} + S_{t_2})$

payoff: $g((S_{t_i})_{i=0,1,2}) = (S_T - K_T)^+$

So $K_{uu} = A_{uu} = 104.05$

$K_{ud} = A_{ud} = 101.28$

$K_{du} = A_{du} = 98.61$

$K_{dd} = A_{dd} > d^2 S_0$

$\Rightarrow \begin{cases} f_{uu} = 0 \\ f_{ud} = 0 \\ f_{du} = 1.23 \\ f_{dd} = 0 \end{cases}$

We use similar approach above to find

$f_u = 4.4$, $f_d = 1.32$

$f = 4.56$

And replicating strategy:

$$\phi_{t_0} = \frac{f_u - f_d}{uS_0 - dS_0} = 0.385$$

$$\phi_{t_1}^{u} = \frac{f_{uu} - f_{ud}}{u^2 S_0 - udS_0} = 0$$

$$\phi_{t_1}^{d} = \frac{f_{du} - f_{dd}}{duS_0 - d^2 S_0} = 0.160$$

Binomial Tree for Exotic Options

Question

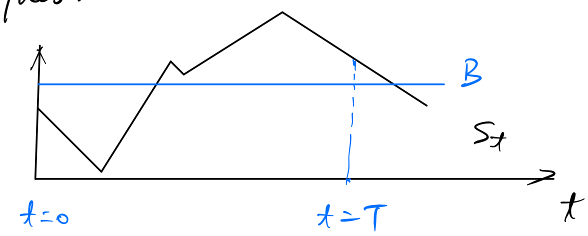
Consider a two step binomial tree with the following parameters:
$S_0 = 100, u = 1.1, d = 0.9$ *and* $r = 0.05$. *Find the prices and the replication strategy of*
(1) A European knock-out call option with a strike price of 95 and a barrier of 90.
(2) A European knock-in call option with a strike price of 95 and a barrier of 90.
(3) A European call option with a strike price 95.

Definition: • A knock-out option is a claim with a predetermined barrier B which is only activated when the barrier is never touched (or crossed) by the underlying stock.

- A knock-in option is a claim with a predetermined barrier B which is only activated when the barrier is touched (or crossed) by the underlying stock.

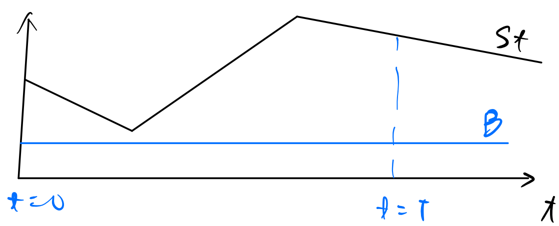
Remark. - Without further information, the activated option is Vanilla.
- The rule could be changed from "touch" to "excess".

Examples:



if knock-out, S_t crossed B, the option is cancelled.

if knock-in, S_t crossed B, the option is activated.



Knock-out is activated

Knock-in is cancelled.

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(1). The payoff function reads:

$$g(S_T) = (S_T - K)\mathbb{1}_{\{\text{the path of } S_t \text{ does not touch or cross } B\}}$$

100 → 110 (19.52) → 121 (26): $f_{uu}^{out} = 26$

110 → 99 (4): $f_{ud}^{out} = 4$

100 → 90 (0) → 99 (0): $f_{du}^{out} = 0$ (it touched $B=90$ at t_1)

90 → 81 (0): $f_{dd}^{out} = 0$ (it touched or crossed $B=90$)

Then $f_u^{out} = 19.52$, $f_d^{out} = 0$ $\Rightarrow$ $f^{out} = 13.94$.

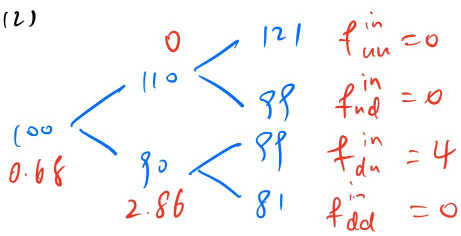
Replicating strategy:

$$\phi_{t_0} = \frac{f_u^{out} - f_d^{out}}{uS_0 - dS_0} = 0.976.$$

$$\phi_{t_1}^{u} = \frac{f_{uu}^{out} - f_{ud}^{out}}{u^2S_0 - udS_0} = 1$$

$$\phi_{t_1}^{d} = \frac{f_{du}^{out} - f_{dd}^{out}}{duS_0 - d^2S_0} = 0$$

(2)



$f_u^{in} = 0$

$f_d^{in} = (1+r\delta t)^{-1}(q \cdot f_{du}^{in} + (1-q) f_{dd}^{in}) = 2.86$

$f^{in} = (1+r\delta t)^{-1}(q f_u^{in} + (1-q) f_d^{in}) = 0.68$

Replicating strategy:

$$\phi_{t_0} = \frac{f_u^{in} - f_d^{in}}{uS_0 - dS_0} \qquad \phi_{t_1}^u = \frac{f_{uu}^{in} - f_{ud}^{in}}{u^2S_0 - udS_0} \qquad \phi_{t_1}^d = \frac{f_{du}^{in} - f_{dd}^{in}}{duS_0 - d^2S_0}$$

$$= -0.143 \qquad\qquad = 0 \qquad\qquad = 0.22$$

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(3). For Vanilla : $f = 14.62$

Note $f = f^{in} + f^{out}$

$$\phi_{t_1}^u = \frac{f_{uu} - f_{ud}}{u^2S_0 - d^2S_0} \qquad \phi_{t_1}^d = \frac{f_{du} - f_{dd}}{duS_0 - d^2S_0} \qquad \phi_{t_0} = \frac{f_u - f_d}{uS_0 - dS_0}$$

$$= 1 \qquad\qquad = 0.22 \qquad\qquad = 0.833$$

One remark also the replicating strategy in direct sum of those from knock-in and knock-out.

Pricing by Martingale Approach

Question

Consider the stock price $(S_t)_{t\geq 0}$ which follows the Black Scholes model. Given risk-free interest rate r, find the price of a financial contract associated with S_t maturing at T with payoff function $g(x) := x^2$.

Solution:
Since S_t follows the Black Scholes Model, we have

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t).$$

Under risk-neutral probability $\mathbb{Q}$, there exists a $B_t^{\mathbb{Q}}$ which is a $\mathbb{Q}$-Brownian motion such that

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t^{\mathbb{Q}})$$

and $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale.

Pricing by Martingale Approach

Denote by V_t the contract price at time $t \leq T$, the price of financial contract with payoff function g. It is clear that $e^{-rt}V_t$ is a martingale under $\mathbb{Q}$. Notice that $V_T = g(S_T) = S_T^2$. Therefore,

$$\begin{aligned}
e^{-rt}V_t &= \mathbb{E}^{\mathbb{Q}}[e^{-rT}V_T|\mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T^2|\mathcal{F}_t] \\
&= S_t^2 e^{-rT}\mathbb{E}^{\mathbb{Q}}[\exp((r-\sigma^2/2)(T-t)+\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))^2|\mathcal{F}_t] \\
&= S_t^2 e^{-rT+(2r-\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}[\exp(2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))|\mathcal{F}_t].
\end{aligned}$$

Note that $B_T^{\mathbb{Q}} - B_t^{\mathbb{Q}}$ is independent of $\mathcal{F}_t$ with distribution $N(0, T-t)$ under $\mathbb{Q}$, then by characteristic function, we have

$$\mathbb{E}^{\mathbb{Q}}[\exp(2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}}))|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[e^{2\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}})}] = e^{2\sigma^2(T-t)}.$$

Finally, $V_t = S_t^2 e^{(r+\sigma^2)(T-t)}$. Taking $t=0$, we have $V_0 = S_0^2 e^{(r+\sigma^2)T}$.