MMAT5010 2223 Test

Q1.(i) Let (x_n) be a sequence in M and x be the weak limit of (x_n). Suppose $x \notin M$. Since M is a norm-closed subspace of X, there is $f \in X^*$ such that $f(M) \equiv 0$ and $f(x) \neq 0$. But since x is the weak limit of (x_n), $f(x) = \lim_{n\to\infty} f(x_n) = 0$. It is a contradiction. Therefore, $x \in M$.

(ii) Consider $X = c_0$ and (e_n). Notice that $c_0^* = \ell^1$. Hence for any $f \in c_0^*$ (denote $f = (f_1, f_2, ..., f_n, ...)$), $\sum_{n=1}^{\infty} |f_n| = \|f\|_{\ell^1} < \infty$. Hence $f(e_n) = f_n \to 0$ as $n \to \infty$. Therefore, (e_n) is weakly convergent but not convergent in c_0.

(iii) Let $(x_n) \subset X$ be weakly convergent to $x \in X$. Let $\dim X = m$ and $\{e_1, ..., e_m\}$ be a base of X. Then there exist $\{\alpha_n(i)\}$ and $\{\alpha(i)\}$ such that $x_n = \sum_{i=1}^m \alpha_n(i) e_i$ for $n = 1, 2, ...$ and $x = \sum_{i=1}^m \alpha(i) e_i$. And let $\{e_1^*, ..., e_m^*\} \subset X^*$ be defined as $e_i^*(e_j) = \delta_{ij}$ for $i, j = 1, ..., m$. Since (x_n) is weakly convergent to x,

$$\alpha_n(i) = e_i^*(x_n) \to e_i^*(x) = \alpha(i) \text{ as } n \to \infty \text{ for each } i = 1, 2, ..., m.$$

Hence

$$\|x_n - x\| = \|\sum_{i=1}^m (\alpha_n(i) - \alpha(i)) e_i\| \leq \sum_{i=1}^m |\alpha_n(i) - \alpha(i)| \|e_i\| \to 0.$$

Therefore, (x_n) is norm convergent to x.

Q2. (i) We claim that T is unbounded. In fact, we take $e_k = (0, 0, ..., 0, 1, 0, ...)$($k$-th entry is 1, others are 0), then $e_k \in c_{00}$ and $\|e_k\|_\infty = 1$. Notice that $\|Te_k\|_\infty = k$ for each $k = 1, 2, ...$ Hence T is unbounded.
(ii) $T^{-1}y(k) = \frac{1}{k} y(k)$ for $y \in c_{00}, k = 1, 2, ...$ Then

$$\|T^{-1}y\|_\infty = \max\{\frac{1}{k}|y(k)| : k = 1, 2, ...\} \leq \max\{|y(k)| : k = 1, 2, ...\} = \|y\|_\infty.$$

Hence T^{-1} is bounded. We claim that $\|T^{-1}\| = 1$. In fact, we take $y_0 = (1, 0, ...)$(the first entry is 1, others are 0). Then $\|y_0\|_\infty = 1$ and $\|T^{-1}y_0\|_\infty = 1$. Therefore, $\|T^{-1}\| = 1$.

Q3.(i) Let $\dim Z = n$ and $\{e_1, ..., e_n\}$ be a base of Z. Then for any $x \in Z$, there exist $\{\alpha_i\}_{i=1}^n$ such that $x = \sum_{i=1}^n \alpha_i e_i$. Since Z is of finite dimension, all norms are equivalent. Hence, there eixsts a constant c such that $\sum_{i=1}^{n} |\alpha_i| \leq c\|x\|$. Therefore, for any linear map T on Z,

$$\|Tx\| = \|\sum_{i=1}^{n} \alpha_i T e_i\| \leq \sum_{i=1}^{n} |\alpha_i| \|Te_i\| \leq (\max_{1 \leq i \leq n} \|Te_i\|) c \|x\|.$$

Therefore, T is bounded.

(ii)Suppose T is bounded, then T is continuous. Hence for any sequence $(x_n) \subset \ker T$ such that $x_n \to x$, we have $Tx = \lim_{n\to\infty} Tx_n = 0$. Hence $x \in \ker T$. Therefore, $\ker T$ is closed.

Now assume that $\ker T$ is closed. Then $X/\ker T$ becomes a normed space. Also, it is known that there is a linear injection $\tilde{T} : X/\ker T \to Y$ such that $T = \tilde{T} \circ \pi$, where $\pi : X \to X/\ker T$ is the natural projection. Since $\dim Y < \infty$ and $\tilde{T}$ is injective, $\dim(X/\ker T) < \infty$. This implies that $\tilde{T}$ is bounded by (i). Hence T is bounded because $T = \tilde{T} \circ \pi$ and π is bounded.