MMAT5010 2223 Assignment 6

Q1. For any $(x, y) \in X \times Y$, $\|\pi(x, y)\| = \|x\| \le \max(\|x\|, \|y\|) = q(x, y)$. Hence π is bounded. We claim that $\|\pi\| = 1$. In fact, we take $x_0 \in X$ such that $\|x_0\| = 1$ and $y_0 = 0_Y$, then $q(x_0, y_0) = 1$ and $\|\pi(x_0, y_0)\| = 1$. Therefore, $\|\pi\| = 1$.

Q2. (i)We claim that T is unbounded. In fact, we take $e_k = (0, 0, ..., 0, 1, 0, ...)$($k$-th entry is 1, others are 0), then $e_k \in X$ and $\|e_k\|_2 = 1$. Notice that $\|Te_k\|_2 = k$ for each $k = 1, 2, ...$ Hence T is unbounded.
(ii) $T^{-1}y(n) = \frac{1}{n}y(n)$ for $y \in \ell_2, n = 1, 2, ...$ Then

$$\|T^{-1}y\|_2 = \sqrt{\sum \frac{1}{n^2}|y(n)|^2} \le \sqrt{\sum |y(n)|^2} = \|y\|_2.$$

Hence T^{-1} is bounded. We claim that $\|T^{-1}\| = 1$. In fact, we take $y_0 = (1, 0, 0, ...)$(the first entry is 1, others are 0). Then $\|y_0\|_2 = 1$ and $\|T^{-1}y_0\|_2 = 1$. Therefore, $\|T^{-1}\| = 1$.