MMAT5010 2223 Assignment 5

Q1. (i)Let $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_\infty)$ be defined by $Tf(x) = \int_a^x f(t)\,dt$. Then

$$\|Tf\|_\infty = \sup_{x\in[a,b]} |Tf(x)| \le \sup_{x\in[a,b]} \int_a^x |f(t)|\,dt \le \int_a^b |f(t)|\,dt = \|f\|_1.$$

Therefore $\|T\| \le 1$. Furthermore, if we let $f : [a,b] \to \mathbb{R}$ to be $f(x) \equiv \frac{1}{b-a}$, then $\|f\|_1 = 1$ and

$$Tf(x) = \frac{x-a}{b-a}.$$

We have $\|Tf\|_\infty = 1$. Hence $\|T\| = 1$.

(ii)Let $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_1)$ be defined by $Tf(x) = \int_a^x f(t)\,dt$. Then

$$\|Tf\|_1 = \int_a^b |Tf(t)|\,dt \le \int_a^b \int_a^t |f(s)|\,ds\,dt \le \int_a^b \int_a^b |f(s)|\,ds\,dt = (b-a)\|f\|_1.$$

Therefore $\|T\| \le b-a$. We claim that $\|T\| = b-a$ by finding a sequence (f_n) in X with $\|f_n\|_1 = 1$ and $\|Tf_n\|_1 \to b-a$. Our f_n is defined by the followings:

- $f_n = 0$ on $[a + \frac{1}{n}, b]$
- $f_n(a) = 2n$
- f_n is a straight line on $[a, a + \frac{1}{n}]$.

It is easy to check that $\|f_n\|_1 = 1$ and $Tf_n(x) = 1$ for $x \in [a + \frac{1}{n}, b]$. Thus $\|Tf_n\|_1 \ge b - (a + \frac{1}{n})$ for every n. Hence f_n is the desired sequence and $\|T\| = b-a$.

Q2. Let $x, y \in X$ such that $\|x-y\| > c > 0$. By Hahn Banach Theorem, there exists $f \in X^*$ such that $f(x-y) = \|x-y\| > c$. Hence $f(x) = f(x-y) + f(y) > c + f(y)$.

Q3. Firstly, we show that T is isometric. $\|Tz(w)\| = \|\sum_{k=1}^n z_k w_k\| \le \|z\|\|w\|$. Hence, $\|Tz\| \le \|z\|$. And by taking $w = \bar{z}$, we have $\|Tz(w)\| = \|z\|^2$. Hence $\|Tz\| = \|z\|$. Therefore, T is isometric. Since T is isometric, T is injective.

Now we show that T is surjective. Let $(e_i)_{i=1}^m$ be the standard base for $\mathbb{C}^m$, i.e. $e_i = (0, 0, ..., 1, 0, ..., 0)$($i-$th entry is 1, others are 0.) Let e_i^* be defined as $e_i^*(e_j) = \delta_{ij}$, then e_i^* is a base for $(\mathbb{C}^m)^*$. Then for any $\phi \in (\mathbb{C}^m)^*$, there exists $(\alpha_i)_{i=1}^m \subset \mathbb{C}$ such that $\phi = \sum_{i=1}^m \alpha_i e_i^*$. So for any $w \in \mathbb{C}^m$, $\phi(w) = \sum_{i=1}^m \alpha_i w_i$. Hence $\phi = T\alpha$, where $\alpha = (\alpha_1, ..., \alpha_m)$. Therefore, T is surjective. Since T is isometic and bijective, T is also bicontinuous. Therefore, T is isometric isomorphic.