MMAT5010 2223 Assignment 4

Q1. For each $x \in \mathbb{R}^n$,

$$\|x\|_\infty = \max_{1\le k\le n} |x_k| \le \sqrt{\sum_{k=1}^{n} |x_k|^2} \le \sqrt{n} \max_{1\le k\le n} |x_k| = \sqrt{n}\|x\|_\infty.$$

Therefore, the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

Q2. Recall that

$$\|T\| = \sup\{\|Tx\| : x \in \mathbb{R}^2, \|x\| = 1\}.$$

Note if $x = (x_1, x_2) \in \mathbb{R}^2$, then $Tx = (x_1 + 2x_2, 3x_2)$. Thus

$$\|Tx\| = \max(|x_1 + 2x_2|, |3x_2|) \le \max(|x_1| + 2|x_2|, 3|x_2|) \le 3\|x\|.$$

This shows $\|T\| \le 3$. We guess that $\|T\| = 3$, and to show it, we want to find $x \in \mathbb{R}^2, \|x\| = 1$ and $\|Tx\| = 3$. We may take $x = (0, 1)$ or $(1, 1)$.

Q3. To show T is discontinuous it needs to show that T is unbounded on the ball $B_{c_{00}}$. Let $e_k = (0, 0, ..., 0, 1, 0, ...)$($k$-th entry is 1, others are 0). Note that $e_k \in B_{c_{00}}$, and $Te_k = ke_k, \|Te_k\| = k$. Hence $\|T\|$ cannot be bounded on $B_{c_{00}}$.