① Let $V = M_{n\times n}(F)$. For $A, B \in V$.

$\langle A, B \rangle := tr(B^* A)$ where $B^* = \overline{B}^T$

Show that $\langle \cdot, \cdot \rangle$ is an inner product.

- $\langle \cdot, \cdot \rangle$ linear in the first argument.

$\forall A_1, A_2, B \in V \quad \forall c \in F$

$\langle cA_1 + A_2, B \rangle$

$= tr(B^*(cA_1 + A_2)) = tr(c \cdot B^* A_1 + B^* A_2)$

$= c \cdot tr(B^* A_1) + tr(B^* A_2) = c \cdot \langle A_1, B \rangle + \langle A_2, B \rangle$

- Symmetric Conjugate. $\overline{\langle A, B \rangle} = \langle B, A \rangle$

$\overline{\langle A, B \rangle} = \overline{tr(B^* A)} = tr(\overline{B^* A}) = tr(\overline{\overline{B}^T A})$

$= tr(B^T \overline{A}) = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ji} \cdot \overline{A}_{ji}$

$= tr(\overline{A}^T B) = \langle B, A \rangle$

- Positive definite.

$\forall A \in V$

$\langle A, A \rangle = tr(A^* A) = tr(\overline{A}^T A)$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{A}_{ji} \cdot A_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ji}|^2$

if $A \neq O_{n\times n}$ then $\langle A, A \rangle > 0$

② V is an inner product space, Then $\forall x, y, z \in V$, $\forall c \in F$

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

(d) $\langle x, x \rangle = 0$ iff $x = 0$

(e) if $\langle x, y \rangle = \langle x, z \rangle$ for $\forall x \in V$. then $y = z$

· linear first arg.

· sym conjugate

· positive def

(a) $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle}$

$= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \overline{\langle cy, x \rangle} = \overline{c \langle y, x \rangle}$

$= \bar{c} \, \overline{\langle y, x \rangle} = \bar{c} \langle x, y \rangle$

(c) $\langle x, \vec{0} \rangle = \langle x, 0 \cdot \vec{0} \rangle = \bar{0} \cdot \langle x, \vec{0} \rangle = 0.$

$\langle \vec{0}, x \rangle = \overline{\langle x, \vec{0} \rangle} = \bar{0} = 0$

(d) $\begin{cases} \langle x, x \rangle > 0 & \text{if } x \neq 0 \\ \langle x, x \rangle = 0 & \text{if } x = 0 \end{cases} \Rightarrow$ "$\langle x, x \rangle = 0 \Leftrightarrow x = 0$"

(e) $\langle x, y \rangle = \langle x, z \rangle \quad \forall x \in V \Leftrightarrow \langle x, y-z \rangle = 0 \quad \forall x \in V$

Let $x = y - z$ then $\langle y-z, y-z \rangle = 0 \Rightarrow y - z = 0$

$y = z$

③ V is inner product space.

norm of $\vec{x} \in V$ is $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle}$

Prop: (a) $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

(b) $\|\vec{x}\| \geq 0$ and $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

(c) $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \cdot \|\vec{y}\|$ Cauchy-schwarz Inequality

(d) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ Triangle Inequality

(c)

- $y = 0$
- $y \neq 0$.

$$0 \leq \|x - cy\|^2 = \langle x - cy, x - cy \rangle$$

$$= \langle x, x \rangle - \bar{c} \langle x, y \rangle - c \langle y, x \rangle + c \cdot \bar{c} \langle y, y \rangle$$

Let $c = \dfrac{\langle x, y \rangle}{\langle y, y \rangle}$

$$0 \leq \langle x, x \rangle - \frac{\overline{\langle x, y \rangle}}{\langle y, y \rangle} \langle x, y \rangle - \frac{\langle x, y \rangle}{\langle y, y \rangle} \cdot \langle y, x \rangle + \frac{\langle x, y \rangle}{\langle y, y \rangle} \frac{\langle y, x \rangle}{\langle y, y \rangle} \langle y, y \rangle$$

$$0 \leq \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2}$$

Hence $|\langle x, y \rangle|^2 \leq \|x\|^2 \cdot \|y\|^2$

(d) $\|x + y\|^2 = \langle x + y, x + y \rangle$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + 2 \cdot \mathrm{Re} \langle x, y \rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2 \cdot |\langle x, y \rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2 \cdot \|x\| \cdot \|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Sec 6.1 Q 15

15. **(a)** Prove that if V is an inner product space, then $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a multiple of the other.

$(\Rightarrow)$ $|\langle x, y\rangle| = \|x\| \cdot \|y\|$

- if $y = \vec{0}$, $y = 0 \cdot x$
- If $y \neq \vec{0}$ let $a = \dfrac{\langle x, y\rangle}{\|y\|^2}$, $z = x - ay$

$$\langle y, z\rangle = \langle y, x - \frac{\langle x, y\rangle}{\|y\|^2} y\rangle$$

$$= \langle y, x\rangle - \frac{\overline{\langle x, y\rangle}}{\cancel{\|y\|^2}} \cancel{\langle y, y\rangle} = 0$$

$$|a|^2 = a \cdot \bar{a} = \frac{\langle x, y\rangle}{\|y\|^2} \cdot \frac{\overline{\langle x, y\rangle}}{\|y\|^2} = \frac{|\langle x, y\rangle|^2}{\|y\|^4}$$

$$= \frac{\|x\|^2 \cdot \|y\|^2}{\|y\|^4} = \frac{\|x\|^2}{\|y\|^2}$$

$$\|x\|^2 = \|z + ay\|^2 = \langle z + ay, z + ay\rangle$$

$$= \|z\|^2 + \|ay\|^2 + \underbrace{\langle z, ay\rangle}_{=0} + \underbrace{\langle ay, z\rangle}_{=0}$$

$$= \|z\|^2 + |a|^2 \cdot \|y\|^2$$

$$= \|z\|^2 + \|x\|^2$$

which implies $\|z\| = 0$, $z = 0$, $x = ay$

$(\Leftarrow)$ trivial

Sec 6.1 Q 23

$x, y \in F^n$

$\langle x, y \rangle := y^* \cdot x$

23. Let $\mathsf{V} = \mathsf{F}^n$, and let $A \in \mathsf{M}_{n \times n}(F)$.

(a) Prove that $\langle x, Ay \rangle = \langle A^*x, y \rangle$ for all $x, y \in \mathsf{V}$.

(b) Suppose that for some $B \in \mathsf{M}_{n \times n}(F)$, we have $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in \mathsf{V}$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathsf{T}(x) = Ax$ and $\mathsf{U}(x) = A^*x$. Show that $[\mathsf{U}]_\beta = [\mathsf{T}]_\beta^*$ for any orthonormal basis β for V.

(a) $\langle x, Ay \rangle = (Ay)^* \cdot x = \bar{y}^T \cdot \bar{A}^T \cdot x = y^* \cdot A^* x = \langle A^* x, y \rangle$

(b) $\langle Bx, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in F^n$, $\{e_1 \cdots e_n\}$ SOB for F^n

let $x = e_i \quad y = e_j$

$B_{ji} = \langle Be_i, e_j \rangle = \langle e_i, Ae_j \rangle = \overline{A_{ij}}$

i.e. $B = A^*$

(c) $\beta = \{Q_1 \cdots Q_n\}$ $\qquad Q = [Q_1 \cdots Q_n]$

$$(Q^* Q)_{ij} = (\bar{Q}^T Q)_{ij} = \bar{Q}_i^T \cdot Q_j = \langle Q_j, Q_i \rangle = \delta_{ij}$$

Thus. $Q^* Q = I_n$

(d) $\beta = \{v_1 \cdots v_n\}$ orthonormal basis

$\forall y \in V \qquad y = \sum_{j=1}^{n} \langle y, v_j \rangle \cdot v_j \qquad [y]_\beta = \begin{pmatrix} \langle y, v_1 \rangle \\ \vdots \\ \langle y, v_n \rangle \end{pmatrix} \in F^n$

For $U(v_j)$ $\qquad U(v_j) = \sum_{i=1}^{n} \langle U(v_j), v_i \rangle \cdot v_i$

$$([U]_\beta)_{ji} = ([U(v_j)]_\beta)_i = \langle U(v_j), v_i \rangle = \langle A^* v_j, v_i \rangle$$

$$= \langle v_j, Av_i \rangle = \langle v_j, T(v_i) \rangle = \overline{\langle T(v_i), v_j \rangle} = \overline{([T]_\beta)_{ij}}$$

i.e $[U]_\beta = [T]_\beta^*$