Definitions. *Two linear operators* T *and* U *on a finite-dimensional vector space* V *are called* ***simultaneously diagonalizable*** *if there exists an ordered basis* β *for* V *such that both* $[\mathsf{T}]_\beta$ *and* $[\mathsf{U}]_\beta$ *are diagonal matrices. Similarly,* $A, B \in \mathsf{M}_{n\times n}(F)$ *are called* ***simultaneously diagonalizable*** *if there exists an invertible matrix* $Q \in \mathsf{M}_{n\times n}(F)$ *such that both* $Q^{-1}AQ$ *and* $Q^{-1}BQ$ *are diagonal matrices.*

$$[T]_\beta = \Lambda_T = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad [U]_\beta = \Lambda_U = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$$

T and U are S.D

share same eigenvectors. but eigen values.

$$T(\beta_1) = \lambda_1 \beta_1 \qquad U(\beta_1) = a_1 \cdot \beta_1$$

See 5.2 Q17.

17. (a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are simultaneously diagonalizable for any ordered basis β.

(b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

(a) T and U are S.D.

$\exists\ \beta'$ basis for V. st $[T]_{\beta'}$ and $[U]_{\beta'}$ are diagonal.

$$\begin{cases} [T]_\beta = [I \circ T \circ I]_\beta = [I]_{\beta'}^{\beta} \cdot [T]_{\beta'} \cdot \underbrace{[I]_{\beta}^{\beta'}}_{Q} = Q^{-1} \cdot [T]_{\beta'}\, Q \\ [U]_\beta = [I \circ U \circ I]_\beta = [I]_{\beta'}^{\beta} \cdot [U]_{\beta'} \cdot \underbrace{[I]_{\beta}^{\beta'}}_{Q} = Q^{-1} \cdot [U]_{\beta'}\, Q \end{cases}$$

(b) $\exists$ invertible Q st $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal.

$Q = [q_1 \cdots q_n]$ $\quad q_i \in F^n$ $\quad \beta := \{q_1 \cdots q_n\}$ basis for F^n

$\gamma = \{e_1 \cdots e_n\}$ standard ordered basis for F^n.

$$\begin{cases} [L_A]_\beta = [I]_\gamma^\beta \cdot [L_A]_\gamma \cdot \underbrace{[I]_\beta^\gamma}_{Q} = Q^{-1} A Q \quad \text{diagonal} \\ [L_B]_\beta = [I]_\gamma^\beta \cdot [L_B]_\gamma \cdot \underbrace{[I]_\beta^\gamma}_{Q} = Q^{-1} B Q \quad \text{diagonal} \end{cases}$$

Sec 5.2 Q 18

18. **(a)** Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., TU = UT).

(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

The converses of (a) and (b) are established in Exercise 25 of Section 5.4.

(a) T and U are S.D.

$\exists\ \beta$ basis for V. s.t. $[T]_\beta$ and $[U]_\beta$ are diagonal.

$\forall v \in \beta.\quad T(v) = \lambda v\ ,\ U(v) = a v$ for some λ and $a \in F$

$$\begin{cases} TU(v) = T(av) = aT(v) = a\lambda v \\ UT(v) = U(\lambda v) = \lambda U(v) = \lambda a v \end{cases}$$

That is TU agrees with UT on basis β.

So $TU = UT$

(b) A and B are S.D.

$\exists$ invertible Q st $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal.

$$\begin{aligned} AB &= Q\,Q^{-1}AQ\,Q^{-1}BQ\,Q^{-1} \\ &= Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} Q^{-1} \\ &= Q \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^{-1} \\ &= Q\,Q^{-1}BQ\ \ Q^{-1}AQ\,Q^{-1} \\ &= BA \end{aligned}$$

Sec 5.4 Q15

15. Use the Cayley–Hamilton theorem (Theorem 5.23) to prove its corollary for matrices. *Warning:* If $f(t) = \det(A - tI)$ is the characteristic polynomial of A, it is tempting to "prove" that $f(A) = O$ by saying "$f(A) = \det(A - AI) = \det(O) = 0$." But this argument is nonsense. Why?

Recall the Cayley-Hamilton Thm :

$T: V \to V$ linear. then $f_T(T) = T_0$

Claim :

$A \in M_{n\times n}(F)$. $f_A(t) = f_{L_A}(t)$ then $f_A(A) = O_{n\times n}$

Proof :

Consider $L_A : F^n \to F^n$
$x \mapsto Ax$

Let β be the standard ordered basis for F^n, then $[L_A]_\beta = A$

Then $f_A(L_A) = f_{L_A}(L_A) = T_0$

CH Thm

So $f_A(A) = [f_A(L_A)]_\beta = O_{n\times n}$

Sec 5.4 Q16

16. Let T be a linear operator on a finite-dimensional vector space V.

(a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

(b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

(a) $f_T(t) = (x-\lambda_1)^{n_1} \cdots (x-a_k)^{n_k}$ $\quad \sum_1^k n_j = \dim(V)$

T-inv. $W \subset V$

$\forall\ T_W, \quad f_{T_W} \mid f_T$

So $f_{T_W} = (x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$ $\quad 0 \le m_j \le n_j$

(b) For any $W \subset V$ $\quad \{0\} \subsetneq W, \quad T(W) \subset W$

f_T splits $\overset{(a)}{\Longrightarrow}$ f_{T_W} splits.

$\exists\ \lambda \in F$ s.t $f_{T_W}(\lambda) = 0$ i.e $\det\left([T_W]_\beta - \lambda I_k\right) = 0$

$[T]_\beta - \lambda I_k$ singular $\Rightarrow$

Then $\exists\ \vec{a} \in F^k$ s.t $([T]_\beta - \lambda I_k)\cdot\vec{a} = \vec{0}$ $\quad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$

Let $\beta = \{w_1 \cdots w_k\}$ $\quad w = a_1 w_1 + \cdots + a_k \cdot w_k \in W$

then $[w]_\beta = \vec{a}$

$[T]_\beta \cdot [w]_\beta = \lambda \cdot [w]_\beta$ i.e. $T(w) = \lambda \cdot w$

Sec 5.4 Q20

20. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then UT = TU if and only if U = $g(\mathsf{T})$ for some polynomial $g(t)$. *Hint:* Suppose that V is generated by v. Choose $g(t)$ according to Exercise 13 so that $g(\mathsf{T})(v) = \mathsf{U}(v)$.

V is T-cylic subspace of itself

$\Leftrightarrow \exists\, v_0 \in V$ st $V = \mathrm{span}(\{v_0, T(v_0), T^2(v_0), \dots\})$

Pf. $UT = TU$

Since $U: V \to V$, $U(v_0) \in V = \mathrm{span}(\{v_0, T(v_0), \dots\})$

So $U(v_0) = a_0 v_0 + a_1 T(v_0) + \dots + a_{k-1} T^{k-1}(v_0)$ for some $a_0 \dots a_{k-1}$

Let $g(t) = a_0 + a_1 t + \dots + a_{k-1} t^{k-1}$

We show that $U = g(T)$

which is equivalent to show $U(T^j(v_0)) = g(T)(T^j(v_0))$ $\forall j \geq 0$

$$\begin{cases} j=0. & U(v_0) = g(T)(v_0) \\ j>0 & U(T^j(v_0)) = U \circ T \circ T^{j-1}(v_0) \\ & = T \circ U \circ T^{j-1}(v_0) \\ & \vdots \\ & = T^j U(v_0) \\ & = T^j \circ g(T)(v_0) \\ & = g(T) \circ T^j(v_0) \\ & = g(T)(T^j(v_0)) \end{cases}$$

U agrees with $g(T)$ on $\{v_0, T(v_0), \dots\}$

So $U = g(T)$

Sec 5.4 Q27

Definition. *Let* T *be a linear operator on a vector space* V, *and let* W *be a* T*-invariant subspace of* V. *Define* $\overline{\mathsf{T}}\colon \mathsf{V}/\mathsf{W} \to \mathsf{V}/\mathsf{W}$ *by*

$$\overline{\mathsf{T}}(v+\mathsf{W}) = \mathsf{T}(v) + \mathsf{W} \quad \textit{for any } v+\mathsf{W} \in \mathsf{V}/\mathsf{W}.$$

27. **(a)** Prove that $\overline{\mathsf{T}}$ is well defined. That is, show that $\overline{\mathsf{T}}(v+\mathsf{W}) = \overline{\mathsf{T}}(v'+\mathsf{W})$ whenever $v+\mathsf{W} = v'+\mathsf{W}$.

(b) Prove that $\overline{\mathsf{T}}$ is a linear operator on V/W.

(c) Let $\eta\colon \mathsf{V} \to \mathsf{V}/\mathsf{W}$ be the linear transformation defined in Exercise 40 of Section 2.1 by $\eta(v) = v + \mathsf{W}$. Show that the diagram of Figure 5.6 commutes; that is, prove that $\eta\mathsf{T} = \overline{\mathsf{T}}\eta$. (This exercise does not require the assumption that V is finite-dimensional.)

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\eta"'] & \mathsf{V} \arrow[d, "\eta"] \\
\mathsf{V}/\mathsf{W} \arrow[r, "\overline{\mathsf{T}}"] & \mathsf{V}/\mathsf{W}
\end{tikzcd}$$

Figure 5.6

(a) if $v + W = v' + W$

then $v - v' \in W$

Since W is T-inv. $T(v) - T(v') = T(v - v') \in W$

So $T(v) + W = T(v') + W$

i.e. $\overline{T}(v+W) = \overline{T}(v'+W)$

(b)

(c) $\forall v \in V$.

$$\begin{cases} \eta T(v) = \eta(T(v)) = T(v) + W \\ \overline{T}\eta(v) = \overline{T}(\eta(v)) = \overline{T}(v+W) = T(v) + W \end{cases}$$