Sec. 2.6 Q20

20. Let V and W be nonzero vector spaces over the same field, and let $T: V \to W$ be a linear transformation.

(a) Prove that T is onto if and only if T^t is one-to-one.
(b) Prove that T^t is onto if and only if T is one-to-one.

Hint: Parts of the proof require the result of Exercise 19 for the infinite-dimensional case.

$T: V \to W$ $\quad T^*: W^* \to V^*$

(a) $(\Rightarrow)$ suppose T is onto

$\forall w \in W$, $\exists v \in V$ s.t. $T(v) = w$

$\forall g \in N(T^*)$ $\quad gT = T^*(g) = 0$ $\quad$ i.e. $\forall v \in V$, $g(T(v)) = 0$

since T onto. $\forall w \in W$ $\quad g(w) = g(T(v)) = 0$ Hence $g = 0$

That means $N(T^*) = \{0\}$, so T^* is 1-1

$(\Leftarrow)$ if T is not onto. then $R(T) \overset{\text{proper}}{\subset} W$

$\exists$ non zero $g_0 \in W^*$ st $g_0(w) = 0$ $\forall w \in R(T)$ $\quad$ i.e. $g_0 T = 0$

$\exists w' \in W \setminus R(T)$ s.t. $g_0(w') \neq 0$

$T^*(g) = gT = gT + g_0 T = (g+g_0)T = T^*(g+g_0)$

but $g(w') \neq (g+g_0)(w')$ so $g \neq g+g_0$

Hence T^* is not 1-1.

(b) $(\Rightarrow)$ suppose T^* is onto.

$\forall f \in V^*$ $\exists g \in W^*$ s.t. $T^*(g) = f \Rightarrow gT = f$

$\forall x \in N(T)$ we have $f(x) = gT(x) = g(0) = 0$, $\forall f \in V^*$

Then $x = 0$. that means $N(T) = \{0\}$. Hence T is 1-1

($\Leftarrow$) Let β be a basis for V.

T is 1-1. so $T(\beta)$ is L.I.

Extend $T(\beta)$ to $T(\beta) \cup \gamma$ a basis for W.

$$T^*: W^* \to V^*$$
$$g \mapsto g \circ T$$

$\forall f \in V^*$, define $g \in W^*$

$$g(w) = \begin{cases} f \circ T^{-1}(w) & w \in T(\beta) \\ 0 & w \in \gamma \end{cases}$$

Then

$$T^*(g)(v) = g \circ T(v) \overset{w = T(v) \in T(\beta)}{=} f \circ T^{-1} \circ T(v) = f(v) \quad \forall v \in \beta$$

so $T^*(g) = f$. T^* is onto.

Sec. 5.1 Q21

21. Let A and $f(t)$ be as in Exercise 20.

(a) Prove that $f(t) = (A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$. *Hint:* Apply mathematical induction to n.

(b) Show that $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

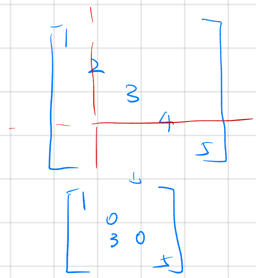
(a) For $n=2$

$$\det(A - tI_2) = \det\left(\begin{bmatrix} A_{11}-t & A_{12} \\ A_{21} & A_{22}-t \end{bmatrix}\right) = (A_{11}-t)(A_{22}-t) - A_{12}A_{21}$$

$q(t) = -A_{12}A_{21}$ is of degree $0 = 2-2 = n-2$

Suppose the case of $n-1$ holds.

For $A \in M_{n\times n}$



$$\det(A - tI_n) = \det\left(\begin{bmatrix} A_{11}-t & & A_{1n} \\ & \ddots & \\ A_{n1} & & A_{nn}-t \end{bmatrix}\right)$$

$$= (A_{nn}-t)\cdot\det\left(\begin{bmatrix} A_{11}-t & & A_{1,n-1} \\ & \ddots & \\ A_{n-1,1} & & A_{n-1,n-1}-t \end{bmatrix}\right)$$

$$+ \sum_{j=1}^{n-1} A_{nj}(-1)^{n+j}\det\left(\underbrace{\begin{bmatrix} A_{11}-t & A_{12} & \cdots & A_{1j-1} & A_{1j+1} & \cdots & A_{1,n-1} \\ A_{21} & A_{22}-t & \cdots & A_{2j-1} & A_{2j+1} & \cdots & A_{2,n-1} \\ \vdots & \vdots & & & & & \\ A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,j-1} & A_{n-1,j+1} & \cdots & A_{n-1,n-1}-t \end{bmatrix}}_{\text{only } n-2 \text{ } t\text{'s in different row and col.}}\right) \quad n-1\times n-1$$

$$= (A_{nn}-t)\,q_n(t) + \sum_{j=1}^{n-1} A_{nj}\,q_j(t)$$

Obviously, $q_j(t) \in P_{n-2}$ and by assumption,

$q_n(t) = (A_{11}-t)\cdots(A_{n-1,n-1}-t) + q_n'(t)$ where $q_n'(t) \in P_{n-3}$

$$\therefore \det(A-tI_n) = (A_{11}-t)\cdots(A_{nn}-t) + \underbrace{(A_{nn}-t)\,q_n'(t) + \sum_{j=1}^{n-1} A_{nj}\,q_j(t)}_{\in P_{n-2}}$$

(b) $f(t) = (A_{11}-t)\cdots(A_{nn}-t) + \underbrace{q(t)}_{\in P_{n-2}}$

$$= (-1)^n t^n + (-1)^{n-1}(A_{11}+\cdots+A_{nn})t^{n-1} + \underbrace{p(t)}_{\in P_{n-2}} + \underbrace{q(t)}_{\in P_{n-2}}$$

$$\therefore a_{n-1} = (-1)^{n-1}(A_{11}+\cdots+A_{nn}) = (-1)^{n-1}\operatorname{tr}(A)$$