1. Given $\beta = \{1,\ 1+x,\ 1+x^2,\ 1+x^3\}$ a basis for $P_3(\mathbb{R})$ and, $\gamma = \{e_{11}\ e_{12}\ e_{21}\ e_{22}\}$, the standard basis for $M_{2\times 2}(\mathbb{R})$

Let $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be defined by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(1) \end{pmatrix}$$

Find $[T]_\beta^\gamma$, $N(T)$

Solution.

$$\begin{cases} T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0\cdot e_{11} + 2\cdot e_{12} + 0\cdot e_{21} + 0\cdot e_{22} \\ T(1+x) = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} = 1\cdot e_{11} + 4\cdot e_{12} + 0\cdot e_{21} + 0\cdot e_{22} \\ T(1+x^2) = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} = 0\cdot e_{11} + 4\, e_{12} + 0\cdot e_{21} + 2\cdot e_{22} \\ T(1+x^3) = \begin{pmatrix} 0 & 4 \\ 0 & 6 \end{pmatrix} = 0\cdot e_{11} + 4\, e_{12} + 0\cdot e_{21} + 6\cdot e_{22} \end{cases}$$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$N(T) = \{ f \in P_3(\mathbb{R}) : T(f) = 0 \}$$
$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 : T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = 0 \}$$

$$0 = T(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$$
$$= T\big((a_0 - a_1 - a_2 - a_3)\cdot 1 + a_1(1+x) + a_2(1+x^2) + a_3(1+x^3) \big)$$
$$= (a_0 - a_1 - a_2 - a_3)\, T(1) + a_1\cdot T(1+x) + a_2\cdot T(1+x^2) + a_3\cdot T(1+x^3)$$
$$= \begin{pmatrix} 0 & 2(a_0-a_1-a_2-a_3) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 4a_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4a_2 \\ 0 & 2a_2 \end{pmatrix} + \begin{pmatrix} 0 & 4a_3 \\ 0 & 6a_3 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & 2(a_0+a_1+a_2+a_3) \\ 0 & 2a_2 + 6a_3 \end{pmatrix}$$

So $\begin{cases} a_1 = 0 \\ a_0 + a_1 + a_2 + a_3 = 0 \\ a_2 + 3a_3 = 0 \end{cases}$ let $a_3 = t$ be the arbitrary variable

Then $\begin{cases} a_0 = 2t \\ a_1 = 0 \\ a_2 = -3t \\ a_3 = t \end{cases}$

$N(T) = \{ t \cdot (2 + 0 \cdot x - 3 \cdot x^2 + x^3) : t \in \mathbb{R} \}$

$\dim(N(T)) = 1$

2. Sec 2.2 Q16

16. Let V and W be vector spaces such that $\dim(\mathsf{V}) = \dim(\mathsf{W})$, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[\mathsf{T}]_\beta^\gamma$ is a diagonal matrix.

Proof: we first choose a basis $\{u_1, \cdots, u_k\}$ for $N(T)$ and extend the basis to a basis $\beta = \{u_1, \cdots, u_k, u_{k+1}, \cdots, u_n\}$ for V.

By dim thm, $\dim(V) = \dim(N(T)) + \dim(R(T))$

$$\dim(R(T)) = \dim(V) - \dim(N(T)) = n-k$$

Besides $R(T) = \text{span}(\{T(u_1), T(u_2), \cdots, T(u_n)\})$ $\quad T(u_1) = \cdots = T(u_k) = \vec{0}$

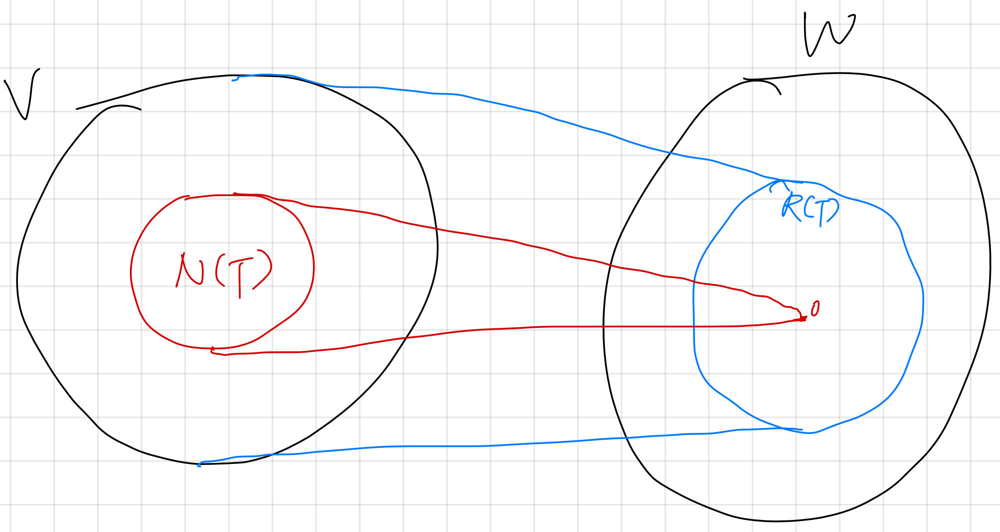
$$= \text{span}(\{T(u_{k+1}), \cdots, T(u_n)\})$$

Therefore, $\{T(u_{k+1}), T(u_{k+2}), \cdots, T(u_n)\}$ is a basis for $R(T)$

we extend it to a basis $\gamma = \{v_1, \cdots, v_k, T(u_{k+1}), \cdots, T(u_n)\}$ for W

Then,

$$[T]_\beta^\gamma = \begin{bmatrix} O_{k\times k} & O_{k\times n-k} \\ O_{n-k\times k} & I_{n-k} \end{bmatrix}$$ is a diagonal matrix



3. Sec 2.3. Q16

16. Let V be a finite-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear.

(a) If $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$ (see the exercises of Section 1.3).

(b) Prove that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$ for some positive integer k.

(a) By rank-nullity thm, we have

$$\text{nullity}(T) = \dim(V) - \text{rank}(T) = \dim - \text{rank}(T^2) = \text{nullity}(T^2)$$

Since $N(T) \subset N(T^2)$, we have that $N(T) = N(T^2)$

Now, for any $v \in R(T) \cap N(T)$, $\exists\, u \in V$ s.t. $v = T(u)$

then $T^2(u) = T(T(u)) = T(v) = 0$, thus $u \in N(T^2)$

Since $N(T^2) = N(T)$, we have $v = T(u) = 0$

Thus $R(T) \cap N(T) = \{0\}$

Besides, $\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$

$$= \dim(V) - 0$$

Thus $R(T) + N(T) = V$, so $V = R(T) \oplus N(T)$

(b) for any $m \geq 0$, $R(T^{m+1}) \subset R(T^m)$

so $\text{rank}(T^m) \geq \text{rank}(T^{m+1})$

$\text{rank}(T) \geq \text{rank}(T^2) \geq \cdots \geq \text{rank}(T^k) \geq \cdots \geq 0$

since $\text{rank}(T) < \infty$, then only finite "$\geq$" above can be strict

So there exist k, such that $\text{rank}(T^k) = \text{rank}(T^{k+j})$ for any $j \geq 0$

Thus $\text{rank}(T^k) = \text{rank}(T^{k+k}) = \text{rank}((T^k)^2)$

By (a), we know $V = R(T^k) \oplus N(T^k)$

4. Sec 2.3 Q12

12. Let V, W, and Z be vector spaces, and let T: V → W and U: W → Z be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.

(a) for any $x \in V$ st $T(x)=0$

then $UT(x) = U(T(x)) = U(0) = 0$

Since UT is one-to-one, $N(UT) = \{0\}$

so $x=0$, which means $N(T) = \{0\}$ i.e T is one-to-one

But U need not to be one-to-one. an example is

$T: \mathbb{R} \to \mathbb{R}^2$ $\quad$ $U: \mathbb{R}^2 \to \mathbb{R}$

$x \mapsto (x,0)$ $\quad$ $(x,y) \mapsto x$

then $UT = I_{\mathbb{R}}$ is one-to-one but U is not one-to-one.

(b) UT is onto, so for any $z \in Z$, $\exists\, v \in V$ s.t. $UT(v) = z$

so there exist $w = T(v) \in W$ s.t. $U(w) = U(T(v)) = UT(v) = z$

That means U is onto.

The example in (a) shows that T need not to be onto

(c) U and T are both onto and one-to-one

$\forall z \in Z$ $\quad \exists\, w \in W$ st. $z = U(w)$

for $w \in W$, $\exists\, v \in V$ st. $w = T(v)$

thus $z = U(w) = U(T(v)) = UT(v)$ thus UT is onto

For $UT(x)=0$, $U(T(x)) = 0$.

Since U is one-to-one. we have $T(x) = 0$

Since T is one-to-one we have $x = 0$

That means $N(UT) = \{0\}$. and UT is one-to-one