13. Let V be a vector space over a field of characteristic not equal to two.

(a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

(b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

Sec 1.5 Q13

This question shows how field influences vector space by linear independence.

Let V be a vector space over field $\mathbb{F}$, then the characteristic of $\mathbb{F}$,

char($\mathbb{F}$) $=2 \iff 2=1+1=0 \iff$ There's no multiplicative inverse $\frac{1}{2}$ for 2

(a) similar to (b)

(b) ($\Rightarrow$) if $\{u, v, w\}$ lin. ind.

Consider the eqn: $a(u+v)+b(u+w)+c(v+w)=\vec{0}$ ✱ for $a, b, c \in \mathbb{F}$

By rearrangement, we have $(a+b)u+(a+c)v+(b+c)w=\vec{0}$

Since $\{u, v, w\}$ is lin. ind. we have $a+b=a+c=b+c=0$

Besides, char($\mathbb{F}$) $\neq 2$ induces that $2 \neq 0$ and $\frac{1}{2}$ exists in $\mathbb{F}$

So $a=\dfrac{(a+b)+(a+c)-(b+c)}{2}=\dfrac{0+0-0}{2}=0$. Similarly, $b=c=0$,

✱ has only one solution $a=b=c=0$. Thus $\{u+v, u+w, v+w\}$ lin. ind.

($\Leftarrow$) Consider the eqn: $au+bv+cw=0$, $a, b, c \in \mathbb{F}$

It can be rewritten as $\frac{a+b-c}{2}(u+v)+\frac{a+c-b}{2}(u+w)+\frac{b+c-a}{2}(v+w)=\vec{0}$

Since $\{u+v, u+w, v+w\}$ lin. ind., the eqn has a unique solution $\frac{a+b-c}{2}=\frac{a+c-b}{2}=\frac{b+c-a}{2}=0$

So $a=\frac{a+b-c}{2}+\frac{a+c-b}{2}=0$, $b=c=0$

Thus $\{u, v, w\}$ is lin. ind.

29. **(a)** Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $\mathsf{W}_1 + \mathsf{W}_2$ is finite-dimensional, and $\dim(\mathsf{W}_1+\mathsf{W}_2) = \dim(\mathsf{W}_1)+\dim(\mathsf{W}_2) - \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$. *Hint:* Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $\mathsf{W}_1 \cap \mathsf{W}_2$ and extend this set to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots v_m\}$ for W_1 and to a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots w_p\}$ for W_2.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $\mathsf{V} = \mathsf{W}_1 + \mathsf{W}_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(\mathsf{V}) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2)$.

Sec 1.6 Q29

This question shows the relation between direct sum and dimension

(a) W_1, W_2 are finite-dim subspaces of V.

Thus $W_1 \cap W_2$ is also a finite-dim subspace of V.

By existence of basis, we can find $\beta_0 = \{u_i\}_{i=1}^k$, a basis for $W_1 \cap W_2$

Then we extend β_0 to be

$\beta_1 = \{u_i\}_{i=1}^k \cup \{v_j\}_{j=1}^m$, a basis for W_1

$\beta_2 = \{u_i\}_{i=1}^k \cup \{w_l\}_{l=1}^p$, a basis for W_2

If $\beta := \beta_1 \cup \beta_2 = \{u_i\}_1^k \cup \{v_j\}_1^m \cup \{w_l\}_1^p$ is a basis for W_1+W_2

Then $\dim(W_1+W_2) = k+m+p = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ ☆

Thus the question reduces to show ☆

1) Since $\beta \subset W_1 + W_2$ and W_1+W_2 is a vector space.

Thus $\text{span}(\beta) \subset W_1+W_2$

2) $\forall x \in W_1+W_2$, $\exists x_1 \in W_1$, $x_2 \in W_2$ s.t. $x = x_1 + x_2$

Since β_1 is a basis for W_1, $\exists a_1 \cdots a_k, b_1 \cdots b_m \in \mathbb{F}$ s.t.

$x_1 = \sum_1^k a_i u_i + \sum_1^m b_j v_j$

Similarly, $\exists c_1 \cdots c_k, d_1 \cdots d_p \in \mathbb{F}$ s.t.

$x_2 = \sum_1^k c_i u_i + \sum_1^p d_l \cdot w_l$

Thus $x = x_1 + x_2 = \sum_1^k (a_i + c_i) u_i + \sum_1^m b_j v_j + \sum_1^p d_l \cdot w_l$

i.e. $x \in \text{span}(\beta)$ and $W_1+W_2 \subset \text{span}(\beta)$

3) Consider the eqn.

$$\sum_1^k a_i u_i + \sum_1^m b_j v_j + \sum_1^p c_\ell \cdot w_\ell = 0 \quad ☆$$

By moving all terms with w_ℓ to the right, we have

$$y := \underbrace{\sum_1^k a_i u_i + \sum_1^m b_j v_j}_{\in W_1} = \underbrace{-\left(\sum_1^p c_\ell \cdot w_\ell \right)}_{\in W_2}$$

So $y \in W_1 \cap W_2 = \text{span}(\beta_0)$. $\exists\, d_1 \cdots d_k \in F$ s.t. $y = \sum_1^k d_i u_i$

$$0 = y - y = \sum_1^k d_i u_i - \left(-\sum_1^p c_\ell \cdot w_\ell\right) = \sum_1^k d_i u_i + \sum_1^p c_\ell w_\ell \quad ☆$$

Since $\beta_2 = \{u_i\}_1^k \cup \{w_\ell\}_1^k$ is lin. ind.

☆ has a unique solution $d_i = c_\ell = 0 \quad \forall i, \ell$.

Thus $y = 0$, which implies $\sum a_i u_i + \sum b_j v_j = 0$

Thus $a_i = b_j = 0 \quad \forall i, j$. by lin. ind. of β_1

Therefore ☆ has a unique solution $a_i = b_j = c_\ell = 0 \quad \forall i, j, \ell$.

So $\beta = \{u_i\} \cup \{v_j\} \cup \{w_\ell\}$ is lin. ind.

By 1), 2), 3), we conclude that β is a basis for $W_1 + W_2$

(b) $(\Rightarrow)$ $V = W_1 \oplus W_2$. Then $W_1 \cap W_2 = \{\vec{0}\}$. and $\dim(W_1 \cap W_2) = 0$

Since $V = W_1 + W_2$, by (a) we have

$$\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$
$$= \dim(W_1) + \dim(W_2)$$

$(\Leftarrow)$ Since $V = W_1 + W_2$, by (a) we have

$$\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

And we also have $\dim(V) = \dim(W_1) + \dim(W_2)$

Therefore $\dim(W_1 \cap W_2) = 0$ i.e. $W_1 \cap W_2 = \{\vec{0}\}$

Thus $V = W_1 \oplus W_2$

34. **(a)** Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

(b) Let $V = R^2$ and $W_1 = \{(a_1, 0) : a_1 \in R\}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

Sec. 1.6 Q34

direct sum

(a) By existence of basis, we can choose a basis $\beta_1 = \{u_i\}_1^k$ for W_1

Then extend it to $\beta = \{u_i\}_1^k \cup \{v_j\}_1^m$, a basis for V.

Let $\beta_2 = \{v_j\}_1^m$, define $W_2 = \text{span}(\beta_2)$

We claim that $V = W_1 \oplus W_2$

1) $\forall\, w \in W_1 \cap W_2$, $\exists\, a_1 \cdots a_k, b_1 \cdots b_m \in \mathbb{F}$ s.t

$$w = \sum_1^k a_i u_i = \sum_1^m b_j v_j$$

Then $0 = w - w = \sum_1^k a_i u_i - \sum_1^m b_j v_j$ ☆

Since β is a basis for V, thus it's lin. ind.

The ☆ has a unique solution $a_i = b_j = 0 \ \forall i, j$

Therefore $w = \vec{0}$ and $W_1 \cap W_2 = \{\vec{0}\}$

2) $V = \text{span}(\{u_1 \cdots u_k, v_1 \cdots v_m\})$

$\overset{☆}{=} \text{span}(\{u_1 \cdots u_k\}) + \text{span}(\{v_1 \cdots v_m\})$

$= W_1 + W_2$

Please show ☆ by yourself.

(b) Any straight line passing through the origin other than the x-axis is a subspace W_2 s.t $\mathbb{R}^2 = W_1 \oplus W_2$

31. Let W be a subspace of a vector space V over a field F. For any $v \in V$ the set $\{v\}+W = \{v+w : w \in W\}$ is called the **coset** of W **containing** v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.
(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{v + W : v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$.

(c) Prove that the preceding operations are well defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in F$.

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the **quotient space of V modulo W** and is denoted by V/W.

Sec 1.3 Q31

Quotient space.

(a) ($\Rightarrow$) If $v+W$ is a subspace of V.

Then $\vec{0} \in v+W$ i.e. $\exists\, w \in W$ s.t. $\vec{0} = v + w$

Thus $v = -w \in W$ since W is a subspace.

($\Leftarrow$) If $v \in W$.

$\forall x \in v+W$, $\exists\, y \in W$ s.t. $x = v+y$

Since $v, y \in W$ and W is a subspace.

$x = v+y \in W$ i.e. $v+W \subset W$

$\forall x \in W$. $x = v + x + (-v)$

$y := x + (-v) \in W$ since W is a subspace.

Thus $\exists\, y \in W$ s.t. $x = v+y \in v+W$ i.e. $W \subset v+W$

In conclusion. $W = v+W$ if $v \in W$, thus $v+W$ is a subspace.