

## Lecture 8

### Invertibility and Isomorphism

Definition: Let  $V$  and  $W$  be vector spaces and let  $T: V \rightarrow W$  be linear. We can say  $T$  is invertible if it is bijective (1-1 and onto) so that  $\exists T^{-1}: W \rightarrow V$  such that:

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \circ T^{-1} = I_W$$

Remark: If  $V$  and  $W$  are of equal finite-dimensions, then  $T: V \rightarrow W$  is invertible iff  $\text{rank}(T) = \dim(V)$ .

$$\overset{\text{"}}{\text{dim}(\text{R}(T))} \overset{\text{"}}{\text{dim}(W)}$$

$T$  is onto

Proposition: The inverse  $T^{-1}: W \rightarrow V$  of an invertible linear transformation  $T: V \rightarrow W$  is linear.

Proof: Let  $\vec{y}_1, \vec{y}_2 \in W$  and  $c \in F$ .  
 $\because T$  is bijective  $\therefore \exists! \vec{x}_1 \in V$  and  $\vec{x}_2 \in V$  such that  
 $T(\vec{x}_1) = \vec{y}_1$  and  $T(\vec{x}_2) = \vec{y}_2$

$$\begin{aligned} \text{So, } T^{-1}(c\vec{y}_1 + \vec{y}_2) &= T^{-1}(cT(\vec{x}_1) + T(\vec{x}_2)) \\ &= T^{-1}(T(c\vec{x}_1 + \vec{x}_2)) \\ &= c\vec{x}_1 + \vec{x}_2 \\ &= cT^{-1}(\vec{y}_1) + T^{-1}(\vec{y}_2) \end{aligned}$$

$\therefore T^{-1}$  is linear.

Example: 1. Let  $A \in M_{n \times n}(F)$  is invertible.

Then:  $L_A: F^n \rightarrow F^n$  defined by  $L_A(\vec{x}) = A\vec{x}$ .

is invertible and the inverse of  $L_A$  is:

$$(L_A)^{-1} = L_{A^{-1}}$$

2. If  $T: V \rightarrow W$  and  $U: W \rightarrow Z$  are invertible linear transformations, then:  $U \circ T$  is also invertible and

$$(U \circ T)^{-1} = T^{-1} U^{-1}$$

$$\left( \underbrace{T^{-1} U^{-1} \circ U \circ T}_{\text{Id}} \right)$$

## Lecture 6

Lemma: Suppose  $T: V \rightarrow W$  is invertible.

Then:  $\dim(V) < +\infty$  iff  $\dim(W) < +\infty$

Proof: And in this case,  $\dim(V) = \dim(W)$   
Suppose  $\dim(V) = n < +\infty$  and  $\beta = \{\vec{x}_1, \dots, \vec{x}_n\}$  is  
a basis for  $V$ . Then:  $W = R(T) = \text{span}\{T(\beta)\}$

$$\therefore \dim(W) \leq n = \dim(V) < +\infty = \text{span} \underbrace{\{T(\vec{x}_1), \dots, T(\vec{x}_n)\}}_{n \text{ elements}}$$

Apply the same argument to  $T^{-1}$  to show that

$$\dim(V) \leq \dim(W)$$

In particular, if  $\dim(V) < +\infty$  and  $\dim(W) < +\infty$   
then:  $\dim(V) \leq \dim(W)$  and  $\dim(W) \leq \dim(V) \Rightarrow \dim(V) \overset{\dim(W)}{=} \dim(V)$



Remark: If  $T: V \rightarrow W$  is onto,  
(linear)

then:  $\dim(W) \leq \dim(V)$

Proposition: Let  $V$  and  $W$  be finite-dimensional vector spaces with ordered basis  $\beta$  and  $\gamma$  respectively.

Let  $T: V \rightarrow W$  be linear transformation.

Then:  $T$  is invertible iff  $[T]_{\beta}^{\gamma}$  is invertible.

Furthermore, 
$$\underbrace{[T^{-1}]_{\gamma}^{\beta}}_{\text{Matrix representation of } T^{-1}} = \underbrace{\left([T]_{\beta}^{\gamma}\right)^{-1}}_{\text{Inverse of matrix.}}$$

Matrix  
representation  
of  $T^{-1}$

Inverse of matrix.

Proof: Suppose  $T$  is invertible. Then:  $\dim(V) = \dim(W) = n$

Since  $T \circ T^{-1} = I_W$ ,  $I_n = [I_W]_{\gamma} = [T \circ T^{-1}]_{\gamma}$

$$\begin{array}{ccccc} W & \xrightarrow{T^{-1}} & V & \xrightarrow{T} & W \\ \gamma & & \beta & & \gamma \end{array}$$

$$I_n = [T]_{\beta}^{\gamma} [T^{-1}]_{\gamma}^{\beta}$$

Similarly,  $T^{-1} \circ T = I_V$ .  $I_n = [I_V]_{\beta} = [T^{-1} \circ T]_{\beta}$

$$I_n = [T^{-1}]_{\gamma}^{\beta} [T]_{\beta}^{\gamma}$$

$\therefore [T]_{\beta}^{\gamma}$  is invertible and  $([T]_{\beta}^{\gamma})^{-1} = [T^{-1}]_{\gamma}^{\beta}$ .

Conversely, suppose  $A := [T]_{\beta}^{\gamma}$  is invertible. ( $\Rightarrow \dim(V) = \dim(W)$ )

'  $\dim(V) = \dim(W)$

$\therefore$  We only need to show  $T$  is one-to-one.

So, suppose  $T(\vec{x}_1) = T(\vec{x}_2)$

$$\Leftrightarrow [T(\vec{x}_1)]_{\gamma} = [T(\vec{x}_2)]_{\gamma}$$

$$\Rightarrow \underbrace{[T]_{\beta}^{\gamma}}_A [\vec{x}_1]_{\beta} = \underbrace{[T]_{\beta}^{\gamma}}_A [\vec{x}_2]_{\beta}$$

$$\Rightarrow [\vec{x}_1]_{\beta} = [\vec{x}_2]_{\beta} \Rightarrow \vec{x}_1 = \vec{x}_2 \quad //$$



Corollary: Let  $V$  be a finite-dimensional vector space with ordered basis  $\beta$ . Let  $T: V \rightarrow V$  be a linear transformation.

Then:  $T$  is invertible iff  $[T]_{\beta}$  is invertible

Furthermore,  $[T^{-1}]_{\beta} = ([T]_{\beta})^{-1}$ .  $[L_A]_{\beta} \leftarrow$  standard ordered basis

Corollary: Let  $A \in M_{n \times n}(F)$ . Then:  $A$  is invertible iff  $L_A$  is invertible.  $(L_A)^{-1} = L_{A^{-1}}$

$$[L_{A^{-1}}]_{\beta} = ([L_A]_{\beta})^{-1} = A^{-1} = [L_{A^{-1}}]_{\beta}$$

$\Rightarrow L_A$

Definition: Let  $V$  and  $W$  be two vector spaces.

We say  $V$  is **isomorphic** to  $W$  if  $\exists$  an invertible linear transformation  $T: V \rightarrow W$ .

In this case,  $T$  is called an **isomorphism** from  $V$  onto  $W$ .

