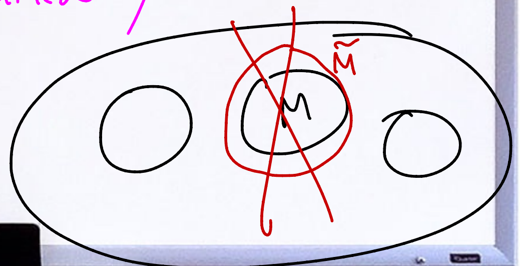
Lecture 4:

Zorn's lemma Assume $\mathcal{C}$ is a collection of subsets of some fixed unnamed set. Assume that for all chain:

$$\underset{\mathcal{C}}{\underset{\cap}{S_1}} \subset \underset{\mathcal{C}}{\underset{\cap}{S_2}} \subset \cdots \quad \text{in } \mathcal{C}, \text{ we have } \bigcup_i S_i \in \mathcal{C}$$

Then: $\mathcal{C}$ contains a maximal element.

(That is, $\exists\, M \in \mathcal{C} \ni$ there is no set $\tilde{M} \in \mathcal{C}$ satisfying: $M \subset \tilde{M}$
$\uparrow$
properly contained)



Theorem: Every vector space has a basis.

Proof: Let $\mathcal{C}$ be the collection of all linearly independent subsets of V.

For any chain $S_1 \subset S_2 \subset \dots \subset S_n \subset \cdots$ in $\mathcal{C}$,

$\bigcup_i S_i$ is also linearly independent. $\therefore \bigcup_i S_i \in \mathcal{C}$.

By Zorn's lemma, $\exists$ maximal linearly independent set M.

We claim that $\mathrm{span}(M) = V$.

If not, $\exists \vec{v} \in V \ni \vec{v} \notin \mathrm{Span}(M)$.

Then: $M \cup \{\vec{v}\}$ is linearly independent.

But $M \subset M \cup \{\vec{v}\}$. Contradition to Zorn's lemma.

$\therefore$ ① $\mathrm{Span}(M) = V$

② M is L.I.

$\Rightarrow$ M is a basis.

Example: Consider $F^{\infty} = \{(a_1, a_2, \dots) : a_j \in F\}$.

Let $S_i = \{\underset{(1,0,\dots 0)}{\vec{e}_1}, \underset{(0,1,0,\dots)}{\vec{e}_2}, \dots, \vec{e}_i\}$

Then: $S_1 \subset S_2 \subset \dots \subset S_i \subset \dots$

Let $S = \bigcup_i S_i$, which is linearly independent.

Obviously $\text{span}(S) \neq F^{\infty}$.

So, we can find $\vec{v} \neq \text{span}(S) \ni S \cup \{\vec{v}\}$ is linearly independent

We can repeat the process.

Question: will the process stop??

Linear Transformation

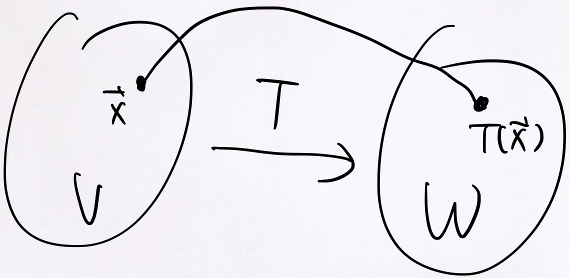
Definition: Let V and W be vector spaces over F.

A linear transformation from V to W is a map $T: V \to W$

such that: (a) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(b) $T(a\vec{x}) = aT(\vec{x})$

for all $\vec{x}, \vec{y} \in V$, $a \in F$.



Proposition: Let $T: V \to W$ be a linear transformation. Then:

(i) $T(\vec{0}_V) = \vec{0}_W$

(ii) $T\left(\sum_{i=1}^{n} a_i \vec{x}_i\right) = \sum_{i=1}^{n} a_i T(\vec{x}_i)$ $\forall\, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in V$

$a_1, a_2, \ldots, a_n \in F$.

(T preserves linear combination)

(i) $T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$

$\Rightarrow$ $T(\vec{0}_V) = \vec{0}_W$. (Cancellation law)

(ii) Use math. induction (exercise)

Examples: • For any vector spaces V and W, we have :

(a) The zero transformation $T_0 : V \to W$ defined by $T_0(\vec{x}) := \vec{0}_W$ for $\forall \vec{x} \in V$

(b) The identity transformation $I_V : V \to V$ defined by $I_V(\vec{x}) = \vec{x}$ for $\forall \vec{x} \in V$.

• Let $A \in M_{m\times n}(F)$ be a $m\times n$ matrice in F.

Define : $L_A : F^n \to F^m$ as: (F^n = space of col vectors of size n)

$$L_A(\vec{x}) \overset{def}{=} A\vec{x}$$

L_A is called the left multiplication by A.

• $T : M_{m\times n}(F) \to M_{n\times m}(F)$ defined by $T(A) \overset{def}{=} A^t$ (transpose of A)

- $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$ (derivative of f) is a lin. transf.

- Let a and $b \in \mathbb{R}$, $a < b$. Then,

  $T : C(\mathbb{R}) \to \mathbb{R}$ defined by:

  (Space of continuous functions)

$$T(f) \overset{def}{=} \int_a^b f(t)\,dt$$

Null space or Range

Definition: Let V and W be vector spaces and $T: V \to W$ be a linear transformation

Then, the null space (or kernel) of T is defined as:

$$N(T) \overset{\text{def}}{:=} \{ \vec{x} \in V : T(\vec{x}) = \vec{0}_W \} \subset V$$

the range (or image) of T is defined as:

$$R(T) := \{ T(\vec{x}) : \vec{x} \in V \} \subset W$$

e.g. For $I_V : V \to V$, $N(I_V) = \{\vec{0}_V\}$, $R(I_V) = V$
(identity)

For $T_0 : V \to W$, $N(T_0) = V$, $R(T_0) = \{\vec{0}_W\}$
(zero transf)

- $L_A : F^n \to F^m \quad (A \in M_{m\times n}(F))$

$N(L_A) = N(A) =$ null space of A

$R(L_A) = \mathcal{C}(A) =$ col space of A (space of linear combination of col vectors of A)

$$A = \begin{pmatrix} | & | & \cdots & | \end{pmatrix}$$

- For $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ defined by $T(f(x)) = f'(x)$, then:

$N(T) = \{ a_0 \in P_n(\mathbb{R}) : a_0 \in \mathbb{R} \}$

$R(T) = P_{n-1}(\mathbb{R})$